Applications of Exponential Functions in the Modeling of Physical Phenomenon

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Many real-world quantities of interest undergo changes that can be adequately modeled using **exponential functions**. Examples of such quantities include the population of a species, the mass of a radioactive substance, the temperature of a cooling object immersed in a fluid, the concentration of a drug in the bloodstream, the value of an investment, and many others. Exponential functions arise in the modeling of such quantities when one assumes that the quantity of interest **changes at a rate proportional to the current size of the quantity**. Mathematically, if $y(t)$ denotes the amount of the quantity at time t, then to say that $y(t)$ changes at a rate proportional to the current amount $y(t)$ is to say that

$$\frac{dy}{dt} = ky(t). \tag{1}$$

You should think of (1) as an equation where the unknown quantity is the function $y(t)$ (the thing you need to find) and the equation involves the derivative of $y(t)$. For these reasons, (1) is called a **differential equation**. Let's analyze the parts of equation (1):

- The $\frac{dy}{dt}$ part in equation (1) is the instantaneous rate change of $y(t)$ with respect to t, in other words, the derivative of $y(t)$.
- The $ky(t)$ part in equation (1) is a proportion of the current amount of $y(t)$; the constant k is called the **constant of proportionality**.

If $y(t)$ is positive and $k > 0$ then $\frac{dy}{dt} > 0$ and therefore $y(t)$ is increasing, and if $k < 0$ then $\frac{dy}{dt} < 0$ and thus $y(t)$ is decreasing.

To find a formula for a function $y(t)$ that solves equation (1), we rewrite equation (1) as

$$\frac{1}{y(t)}\frac{dy}{dt} = k \tag{2}$$

and this is valid so long as $y(t) \neq 0$. The left-hand side of equation (2) is the derivative of $\ln|y(t)|$:

$$\frac{d}{dt}\ln|y(t)| = \frac{1}{y(t)}\frac{dy}{dt}.$$

Therefore, integrating both sides of equation (2) with respect to t we obtain

$$\int \frac{1}{y(t)}\frac{dy}{dt}\,dt = \int k\,dt$$

$$\Longrightarrow \ln|y(t)| = kt + C.$$

If we take the exponential of both sides of the last equation then

$$e^{\ln|y(t)|} = e^{kt+C}$$

$$\Longrightarrow |y(t)| = e^{kt}e^{C}.$$

Using the definition of the absolute value function, we can rewrite the last equation as

$$y(t) = Ae^{kt}$$

where $A = \pm e^C$ is a constant. We have thus shown that any quantity $y(t)$ that is never zero and satisfies the equation (1) is an exponential function. In many applications, $y(t)$ is a positive quantity such as mass, concentration, temperature, etc., and therefore in that case $A > 0$. If $k > 0$ then $y(t)$ undergoes **exponential growth** and if $k < 0$ then $y(t)$ undergoes **exponential decay**. If we evaluate $y(t)$ at time $t = 0$ we obtain

$$y(0) = Ae^0 = A$$

and therefore A is the initial amount of the quantity $y(t)$ at time $t = 0$.

In Figure 1, three representative graphs of exponential growth are displayed, and in Figure 2, three representative graphs of exponential decay are displayed, all for the case that $A = 1$.

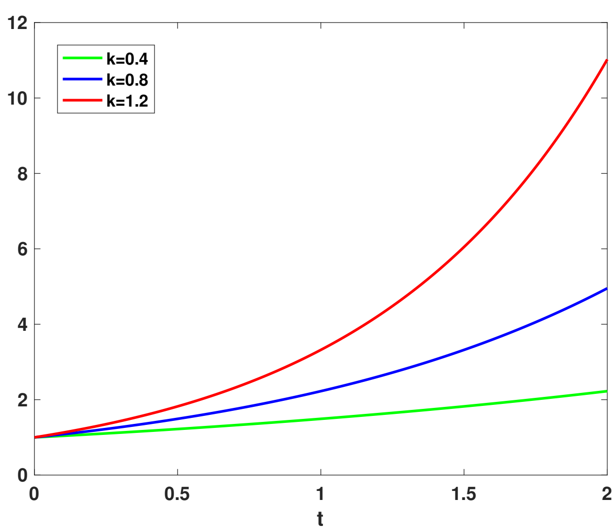

Figure 1: Exponential growth

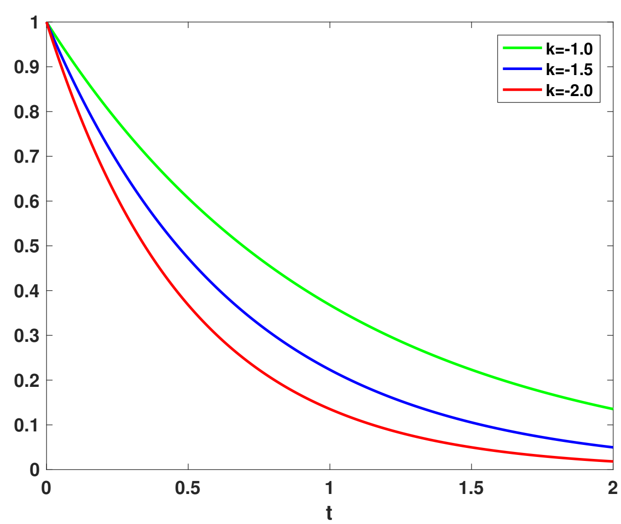

Figure 2: Exponential decay

Example. A bacteria culture growing at a rate proportional to its size, grows from 2000 bacteria to 3000 in 2 hours. Write down a differential equation that represents this situation.

How long will it take for the culture to double in size?

Solution. Let $y(t)$ be the number of bacteria at time t. Then the differential equation modeling this situation is

$$\frac{dy}{dt} = ky(t).$$

From our work above, the solution to the above differential equation is $y(t) = Ae^{kt}$ where A is the initial amount of the bacteria, in other words, $y(0) = 2000$. We are told that the initial value of $y(t)$ is $A = 2000$. Therefore we can write

$$y(t) = 2000\,e^{kt}.$$

We need to find k. To do this, we use that fact that at $t = 2$ hours the amount of bacteria is 3000, in other words, $y(2) = 3000$. We can substitute these values for t and $y(t)$ and solve for k:

$$\begin{aligned} 3000 &= 2000\,e^{2k} \\ \implies \frac{3}{2} &= e^{2k} \\ \implies \ln(3/2) &= 2k \\ \implies k &= \frac{1}{2}\ln(3/2) \end{aligned}$$

Now we know the value of k. To find how long it will take for the culture to double in size we need to find the time t such that $y(t) = 2A$, in other words, when does $y(t) = 4000$? To that end:

$$\begin{aligned} \implies \quad 4000 &= 2000\,e^{1/2\ \ln(3/2)t} \\ \implies \quad 2 &= e^{1/2\ \ln(3/2)t} \\ \implies \quad \ln(2) &= \frac{1}{2}\ln(3/2)t \\ \implies \quad t &= \frac{2\ln(2)}{\ln(3/2)} \end{aligned}$$

Thus, it will take $t = \frac{2\ln(2)}{\ln(3/2)} \approx 3.41$ hours for the culture to double in size. In general, we now show that no matter what the initial value A is, the time to double $y(t)$ from the initial value

A is only dependent on k. To find the doubling-time, we must find t such that $y(t) = 2A$:

$$\begin{aligned} 2A &= A\,e^{kt} \\ \implies 2 &= e^{kt} \\ \implies t &= \frac{\ln(2)}{k} \end{aligned}$$

The initial value A does not appear in this last equation, so t is independent of A, meaning the doubling-time will always be $\frac{\ln 2}{k}$ no matter what A is. ■

Example. The half-life of a substance under decay is the time it takes for the amount of the substance to decrease by 50%. Polonium-214 has a very short half-life of 1.4×10^{-4} seconds. If a Polonium-214 sample has an initial mass of 50 mg, find a formula for the mass that remains after t seconds.

Solution. The half-life is the time it takes for the reduction of a substance, in this case Polonium-214, to half of the original amount. We are told that $y(1.4 \times 10^{-4}) = \frac{A}{2}$ where $A = 50$ mg. Therefore, since $y(t) = Ae^{kt}$ we can write that

$$\begin{aligned} \frac{A}{2} &= A\,e^{k \cdot 1.4 \times 10^{-4}} \\ \implies \frac{1}{2} &= e^{k \cdot 1.4 \times 10^{-4}} \\ \implies \ln(1/2) &= k\,(1.4 \times 10^{-4}) \\ \implies k &= \frac{\ln(1/2)}{1.4 \times 10^{-4}} \\ \implies k &= -\frac{10^4 \ln(2)}{1.4} \end{aligned}$$

Thus, the formula for the mass of Polonium-214 that remains after t seconds is

$$y(t) = 50\,e^{\left(-\frac{10^4 \ln(2)}{1.4}\right)t}$$

The numerical value of k is

$$k \approx -4951.05$$

which is quite high and explains why Polonium-214 decays very rapidly. ■

Carbon dating using radioactive carbon

An important application of exponential decay is in dating events from the Earth's past, and in particular, in dating the death of an organism which in turn could be used to date a natural geological event. During the life of an organism, the ratio of the amount of radioactive carbon-14 (^{14}C) to ordinary carbon in the organism stays fairly constant. However, when the organism dies, ^{14}C is no longer being ingested by the organism and consequently begins to decay at an exponential rate. The half-life of carbon-14 is about 5730 years.

Example. The frozen remains of a young Incan woman were discovered by archeologist Johan Reinhard on Mt. Ampato in Peru during an expedition in 1995. Measurements indicated that 94% of the original carbon-14 at the time of death was present in the remains. What year did the young woman die?

Solution. Let $y(t) = Ae^{kt}$ be the amount of carbon-14 after t years from the death of the young Incan woman, where A is the amount of carbon-14 at the time of death. Since the half-life of carbon-14 is 5730, we have $y(5730) = \frac{1}{2}A$ and therefore

$$\frac{1}{2}A = Ae^{5730k}.$$

Canceling the A and applying ln to both sides we obtain

$$\ln(1/2) = 5730k$$

and thus solving for k we obtain

$$k = -\frac{\ln(2)}{5730}.$$

Now that we have k, we can compute the time t when $y(t)$ is 94% of the original amount A:

$$0.94A = Ae^{-\frac{\ln(2)}{5730}t}$$

$$\implies \ln(0.94) = -\frac{\ln(2)}{5730}t$$

$$\implies t = -5730\frac{\ln(0.94)}{\ln(2)}$$

$$\implies t \approx 511$$

Thus, after 511 years of the young woman's death, the amount of carbon-14 was 94% of the original amount. Therefore, the young woman died in approximately $1995 - 511 = 1484$. ■

Example. After 3 days a sample of radon-222 decayed to 58% of its original amount.
(a) What is the half-life of radon-222?
(b) How long would it take for a sample to decay to 10% of its original amount?

Example. A bacteria culture grows with constant relative growth rate of k. After 2 hours there are 600 cells of the bacteria and after 8 hours there are 75,000 cells.

(a) Find the initial population of the bacteria.

(b) Find the population of the bacteria after 5 hours.

(c) Find the rate of growth of the bacteria after 5 hours.

(d) When will the population reach 200,000?

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Exercise 1. Californium-252 (symbol Cf) is a radioactive isotope discovered in 1950 at the University of California Radiation Laboratory in Berkeley. One microgram of californium-252 emits 170 million neutrons per minute, making it a useful startup source for some nuclear reactors. It has a half-life of about 2.645 years.

(a) Write down a differential equation that models the radioactive decay of Cf.

(b) What is the value of k in the decay equation for Cf?

(c) How long will it take for 95% of a sample of Cf to desintegrate?

(d) Graph the decay function for a sample initially containing 100 micrograms of Cf. Your graph should be properly labelled and should take into consideration your findings in part (b).

Exercise 2. A method to detect art forgery, and in particular forged paintings, is to use carbon dating. Determine the authenticity of a painting that today contains 9.85 grams of carbon-14 that was known to contain 10 grams of carbon-14 in the year 1600. Approximately how old is the painting? Is the painting authentic?