Review Questions on Parametric Curves & Polar Coordinates

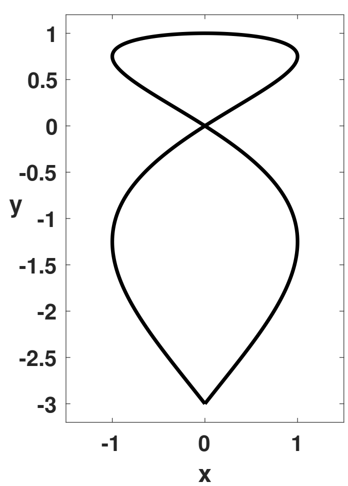
Prof. Cesar Aguilar
Department of Mathematics, SUNY Geneseo

Disclaimer: This is a list of questions to guide you through your studies in Parametric Curves (in 2D) and Polar Coordinates. Not everything that appears in these questions will necessarily appear in the test, and conversely, there might be a question in the test that was not covered by these questions. Use these questions only as a guide in your studies and don't feel like you need to answer every question to be 100% ready for your test. **Solutions to these questions will not be provided.**

Note: When you give a parametrization for a curve, you must also give the interval where the parametrization is defined. Without an interval, a parametrization is (almost) useless.

Parametric Curves

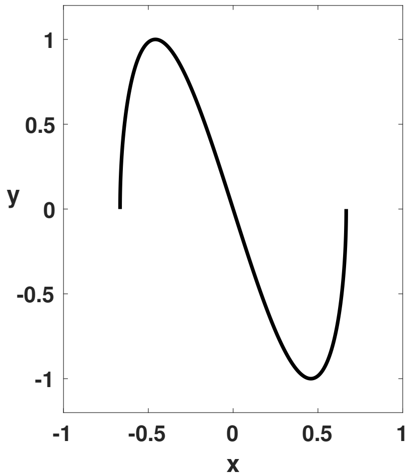
1. Parametrize the line segment starting at the point $(1,-3)$ and terminating at the point $(-4,2)$.
2. Parametrize the half-line with initial point $(2,3)$ and that passes through the point $(-1,-1)$.
3. Parametrize the graph of $y = \sin(x)$ starting at the point $(3\frac{\pi}{2},-1)$ and ending at the point $(0,0)$.
4. Parametrize the ellipse centered at $(3,4)$, moving counterclockwise, starting at the point $(2,4)$ and passing through the point $(3,1)$.
5. Parametrize the circle of radius $r = 3$ centered at $(-3,-3)$ with clockwise orientation and starting at the point $(0,-3)$.
6. The curve shown below has the given parametrization.



$$x(t) = \sin(\pi t)$$
$$y(t) = -t^2 + 2t$$
$$I = [-1,3]$$

   (a) With arrows on the curve, indicate the direction of the parametrization. What is the initial and terminal point of the parametrization?
   (b) Find analytically the point on the curve where the tangent line is horizontal. What is the corresponding t value?
   (c) Find analytically one of the points on the curve where the tangent line is vertical. What is the corresponding t value?
   (d) Find the equation of the tangent line to the curve at the point corresponding to $t = 2$.
   (e) Setup the integral that evaluates to the arc length of the curve but do not attempt to compute the integral (simplify the integrand as much as possible).

7. The curve shown below has the given parametrization.



$$x(t) = \frac{1}{3}t^3 - t$$
$$y(t) = \sin(\pi t)$$
$$I = [-1, 1]$$

(a) With arrows on the curve, indicate the direction of the parametrization. What is the initial and terminal point of the parametrization?
(b) Find analytically the points on the curve where the tangent line is horizontal.
(c) Find the equation of the tangent line to the curve through the origin.
(d) Setup the integral that evaluates to the arc length of the curve but do not attempt to compute the integral (simplify the integrand as much as possible).

8. Consider the parametric curve

$$x(t) = t$$
$$y(t) = \sqrt{1 - t^2}$$
$$I = [-1, 0]$$

(a) Find a Cartesian equation of the curve being parametrized and graph the curve given by the Cartesian equation.
(b) Indicate on the graph which portion of the curve is being parametrized and the orientation of the parametrization. Indicate the initial and terminal point of the parametrization.
(c) Find a parametrization for the same curve but parametrized with opposite orientation.

9. A ring is constrained to move along a wire that has the shape of the graph of a function $y = f(x)$. The trajectory of the ring is given by the parametrization

$$x(t) = \cos(t)$$
$$y(t) = \cos^3(t)$$
$$I = [0, 2\pi]$$

(a) Find $f(x)$ and graph it on the interval $-2 \le x \le 2$.
(b) Indicate on the graph of $f(x)$ the portion of the wire that the ring moves along and give a complete written description of the motion of the ring.
(c) Setup the integral that evaluates to the distance travelled by the ring but do not attempt to compute the integral (simplify the integrand as much as possible).

Polar Coordinates

1. For each case, sketch the region with the given polar coordinates description.

   (a) $1 \le r \le 2, \quad \frac{\pi}{4} \le \theta \le \frac{5\pi}{4}$

   (b) $1 < r \le 2, \quad \frac{\pi}{4} \le \theta < \frac{5\pi}{4}$

   (c) $-\infty < r < \infty, \quad \frac{\pi}{2} \le \theta \le \frac{3\pi}{4}$

   (d) $0 \le r < 1, \quad \frac{\pi}{2} \le \theta \le \frac{3\pi}{4}$

2. What is a polar equation for the line $x = 7$? What about for the line $y = -3$? What about for the line $y = 7x - 3$? In each case, isolate for r as a function of θ.

3. What is a polar equation for the circle $(x-3)^2 + (y+2)^2 = 7$? (Hint: Expand the Cartesian equation first and then substitute for x and y.)

4. What is a polar equation for an ellipse centered at $(-1,3)$, passing through $(-1,0)$ and $(3,3)$? (Hint: What is the Cartesian equation for the ellipse?)

5. For each case, replace the polar equation with an equivalent Cartesian equation, and then use the Cartesian equation to identify (i.e., graph) the curve given by the polar equation.

   (a) $r^2 \sin(2\theta) = 2$

   (b) $r = \csc(\theta) e^{r\cos(\theta)}$

   (c) $r^2 = 4r\sin(\theta)$

   (d) $r\sin(\theta) = \ln(r) + \ln(\cos(\theta))$

   (e) $r = 2\cos(\theta) - \sin(\theta)$

   (f) $\cos^2(\theta) = \sin^2(\theta)$

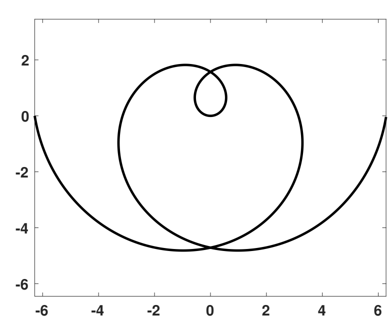
   (g) $r = 4\tan(\theta)\sec(\theta)$

   (h) $r^2 + 2r^2\cos(\theta)\sin(\theta) = 1$

6. Consider the curve with polar equation $r = 1 + \cos(\theta)$, where $0 \le \theta \le 2\pi$.

   (a) By hand, graph the curve by plotting the points corresponding to $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$ and connecting the points by a smooth curve.
   (b) Find the points on the curve where the tangent line to the curve is horizontal.
   (c) Find the equation of the tangent line to the curve when $\theta = \frac{\pi}{6}$.
   (d) Setup the integral that evaluates to the arc length of the curve but do not attempt to compute the integral (simplify the integrand as much as possible).
   (e) Setup the integral that evaluates to the area enclosed by the curve but do not compute the integral.

7. The curve given in polar coordinates by $r = \theta$, for $-2\pi \le \theta \le 2\pi$, is shown below.



(a) Give a parametrization $x(\theta)$ and $y(\theta)$ for the given curve on the interval $-2\pi \leq \theta \leq 2\pi$.
(b) Indicate with arrows on the curve the direction of the parametrization from part (a) and state its initial and terminal points.
(c) What is the point on the curve corresponding to $\theta = \pi$? Draw it on the curve.
(d) Find the equation of the tangent line corresponding to the point $\theta = \pi$. (**Answer:** $y = \pi x - \pi^2$)
(e) Setup the integral that evaluates to the arc length of the curve but do not compute the integral.

8. Find the area of the region enclosed by the cardioid $r = 2(1 + \cos(\theta))$. (**Answer:** 6π).