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README: In many cases, it is important to properly denote the starting value of the index in a series, for example $\sum_{k=0}^{\infty} a_k$ or $\sum_{n=1}^{\infty} a_n$, etc. However, many theorems hold regardless of where the index starts and for this reason we sometimes omit the starting of the index and simply write $\sum a_n$ or $\sum a_k$, etc.

Below is the basic theorem regarding the sum, difference, and constant multiples of convergent series.

BASIC THEOREM. Suppose that the series $\sum a_n$ and $\sum b_n$ are both convergent, and that they converge to $\sum a_n = A$ and $\sum b_n = B$. Then

(i) The series $\sum (a_n + b_n)$ converges to $\sum (a_n + b_n) = \sum a_n + \sum b_n = A + B$

(ii) The series $\sum (a_n - b_n)$ converges to $\sum (a_n - b_n) = \sum a_n - \sum b_n = A - B$

(iii) If c is any constant then the series $\sum ca_n$ converges to $\sum (ca_n) = c \sum a_n = cA$

In words, BASIC THEOREM says the following:

BASIC THEOREM in WORDS.

(i) **The sum of two convergent series is a convergent series.**

(ii) **The difference of two convergent series is a convergent series.**

(iii) **A constant multiple of a convergent series is a convergent series.**

The following observation is useful: If the series $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} b_n$ diverges then both $\sum_{n=1}^{\infty} (a_n + b_n)$ and $\sum_{n=1}^{\infty} (a_n - b_n)$ diverge. Why? Well, if $\sum (a_n + b_n)$ converges then because $\sum a_n$ also converges then the difference $\sum (a_n + b_n) - \sum a_n$ also converges by BASIC THEOREM. But $\sum (a_n + b_n) - \sum a_n = \sum b_n$, and we are given that $\sum b_n$ diverges!

Example 1: Suppose that $\sum a_n = 5$, $\sum b_n = -11$, and $\sum c_n = 200$. Using the BASIC THEOREM, the series

$$\sum (9a_n + 3b_n - 4c_n)$$

is convergent because it is a sum, difference, and constant multiple of convergent series. This series converges to

$$\sum (9a_n + 3b_n - 4c_n) = 9 \sum a_n + 3 \sum b_n - 4 \sum c_n = 9(5) + 3(-11) - 4(200) = -788$$

1. The Geometric Series

A very important series is the Geometric series:

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + r^4 + \cdots$$

We showed that the partial sums of the geometric series are

$$s_n = \frac{1 - r^{n+1}}{1 - r}$$

and therefore if $|r| < 1$ then

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{1 - r^{n+1}}{1 - r} \right) = \frac{1}{1 - r}$$

Thus, the Geometric series converges only when $|r| < 1$ and in this case the series converges to

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}.$$

When $|r| \geq 1$, the Geometric series does not converge.

Example 2: If possible, find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{(\ln 3)^n} = \frac{1}{(\ln 3)} + \frac{1}{(\ln 3)^2} + \frac{1}{(\ln 3)^3} + \cdots$$

Solution: Although this is a Geometric series, the index n begins at $n = 1$ but the formula $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ is valid for when n begins at $n = 0$. To deal with this, we can re-index the series to start at $n = 0$ as follows:

$$\sum_{n=1}^{\infty} \frac{1}{(\ln 3)^n} = \sum_{n=0}^{\infty} \frac{1}{(\ln 3)^{n+1}}$$

Notice that $\sum_{n=0}^{\infty} \frac{1}{(\ln 3)^{n+1}}$ gives the exact same series that we were given:

$$\sum_{n=0}^{\infty} \frac{1}{(\ln 3)^{n+1}} = \frac{1}{(\ln 3)} + \frac{1}{(\ln 3)^2} + \frac{1}{(\ln 3)^3} + \cdots$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{(\ln 3)^n} = \sum_{n=0}^{\infty} \frac{1}{(\ln 3)^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{(\ln 3)^n \ln 3}$$

$$= \frac{1}{\ln 3} \left(\sum_{n=0}^{\infty} \frac{1}{(\ln 3)^n} \right) \qquad \text{here } r = \tfrac{1}{\ln 3} < 1$$

$$= \frac{1}{\ln 3} \left(\frac{1}{1 - \frac{1}{\ln 3}} \right)$$

$$= \frac{1}{\ln 3 - 1} \qquad \text{after simplifying}$$

Example 3: If possible, find the sum of the series

$$\sum_{n=0}^{\infty}(\sqrt{2})^n$$

Solution: This is a Geometric series with $r = \sqrt{2}$. Since $\sqrt{2} > 1$ the series diverges!

Example 4: Find the sum of the series

$$\sum_{n=3}^{\infty}\frac{2^n}{7^n} = \frac{2}{7} + \frac{4}{49} + \frac{8}{343} + \cdots$$

Solution: This is a Geometric series with n starting at $n = 3$. We re-index the series to start at $n = 0$:

$$\sum_{n=3}^{\infty}\frac{2^n}{7^n} = \sum_{n=0}^{\infty}\frac{2^{n+3}}{7^{n+3}}$$

Now we just pull out $\frac{2^3}{7^3}$ from the sum and use the formula for the Geometric series:

$$\begin{aligned}\sum_{n=3}^{\infty}\frac{2^n}{7^n} = \sum_{n=0}^{\infty}\frac{2^{n+3}}{7^{n+3}} = \sum_{n=0}^{\infty}\frac{2^n2^3}{7^n7^3} = \frac{2^3}{7^3}\sum_{n=0}^{\infty}\frac{2^n}{7^n} &= \frac{2^3}{7^3}\sum_{n=0}^{\infty}\left(\frac{2}{7}\right)^n \qquad \text{here } r = \tfrac{2}{7} < 1\\ &= \frac{2^3}{7^3}\left(\frac{1}{1-\frac{2}{7}}\right)\\ &= \frac{8}{245} \qquad \text{after simplifying}\end{aligned}$$

Example 5: Determine if the series $\displaystyle\sum_{n=0}^{\infty}\left(\frac{2^n-1}{3^n}\right)$ converges or diverges.

Solution: We can write this series as

$$\sum_{n=0}^{\infty}\left(\frac{2^n-1}{3^n}\right) = \sum_{n=0}^{\infty}\left(\frac{2^n}{3^n} - \frac{1}{3^n}\right)$$

So the series is the difference $\sum_{n=0}^{\infty}(a_n - b_n)$ where $a_n = \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$ and $b_n = \frac{1}{3^n} = \left(\frac{1}{3}\right)^n$. These are both Geometric series and they converge to

$$\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{1-\frac{2}{3}} = 3$$

and

$$\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}.$$

Therefore, by BASIC THEOREM:

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\frac{2^n-1}{3^n}\right) &= \sum_{n=0}^{\infty}\left(\frac{2^n}{3^n}-\frac{1}{3^n}\right)\\ &= \sum_{n=0}^{\infty}\left(\frac{2^n}{3^n}\right)-\sum_{n=0}^{\infty}\left(\frac{1}{3^n}\right)\\ &= 3-\frac{3}{2}\\ &= \frac{3}{2}\end{aligned}$$

Example 6: Determine if the series $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{3}{2^n}$ converges or diverges.

Solution: First notice that $(-1)^{n+1}=(-1)^n(-1)$. Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{3}{2^n} &= \sum_{n=0}^{\infty}(-1)^n(-1)\frac{3}{2^n}\\ &= -3\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n} && \text{take out constant } 3(-1)\\ &= -3\sum_{n=0}^{\infty}(-\tfrac{1}{2})^n && \text{here } r=-\tfrac{1}{2}\\ &= -3\frac{1}{1-(-\frac{1}{2})}\\ &= -2 && \text{after simplyfing}\end{aligned}$$

2. Testing for Divergence when $\lim_{n\to\infty} a_n \neq 0$

For a series $\sum_{n=1}^{\infty} a_n$ that converges it must be true that the sequence $\{a_n\}$ converges to zero:

$$\lim_{n\to\infty} a_n = 0.$$

Another way of saying this is that if $\lim_{n\to\infty} a_n$ does not equal zero then the series $\sum_{n=1}^{\infty} a_n$ DIVERGES! This is called the **Divergence Test**.

Example 7: Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty}\frac{5^n}{4^n+3}.$$

Solution: Let's see if $\lim_{n\to\infty} a_n \neq 0$:

$$\begin{aligned}
\lim_{n\to\infty} \frac{5^n}{4^n+3} &= \lim_{n\to\infty} \frac{5^n \ln(5)}{4^n \ln(4)} \qquad \frac{\infty}{\infty} \text{ so use L.H.R} \\
&= \frac{\ln(5)}{\ln(4)} \lim_{n\to\infty} \frac{5^n}{4^n} \\
&= \frac{\ln(5)}{\ln(4)} \lim_{n\to\infty} \left(\frac{5}{4}\right)^n \\
&= \infty
\end{aligned}$$

because $\lim_{n\to\infty} \left(\frac{5}{4}\right)^n = \infty$. Therefore, because $\lim_{n\to\infty} \frac{5^n}{4^n+3} \neq 0$, the series $\sum_{n=1}^{\infty} \frac{5^n}{4^n+3}$ diverges.

Example 8: Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{3n^2+6n+1}{11n^2-n+4}$$

Solution: Let's see if $\lim_{n\to\infty} a_n \neq 0$:

$$\lim_{n\to\infty} \left(\frac{3n^2+6n+1}{11n^2-n+4}\right) = \frac{3}{11}$$

So, because $\lim_{n\to\infty} a_n \neq 0$, the series $\sum_{n=1}^{\infty} \frac{3n^2+6n+1}{11n^2-n+4}$ diverges.

Example 9: Determine whether the series converges or diverges.

$$\sum_{k=0}^{\infty} \ln \frac{1}{3^k}$$

Solution: Let's see if $\lim_{k\to\infty} a_k \neq 0$:

$$\lim_{k\to\infty} \ln \frac{1}{3^k} = \ln\left(\lim_{k\to\infty} \frac{1}{3^k}\right) = -\infty$$

because $\lim_{k\to\infty} \frac{1}{3^k} = 0$ and $\lim_{x\to 0^+} \ln(x) = -\infty$. Therefore, the series $\sum_{k=0}^{\infty} \ln \frac{1}{3^k}$ diverges because $\lim_{k\to\infty} a_k \neq 0$.

Example 10: Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} + \frac{1}{2^n}\right)$$

Solution: Let's see if $\lim_{n\to\infty} a_n \neq 0$:

$$\lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{2^n}\right) = \lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} \frac{1}{2^n} = 0 + 0 = 0$$

So, it is true that $\lim_{n\to\infty} a_n = 0$ and thus we cannot conclude that the series diverges (we certainly cannot conclude that it converges). We need to do further analysis. **If** the series converges then because $\sum_{n=1}^{\infty} \frac{1}{2^n}$ also converges (it is a geometric series with $r = 1/2$) then the difference

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} + \frac{1}{2^n}\right) - \sum_{n=1}^{\infty} \frac{1}{2^n}$$

would also converge (by BASIC THEOREM). But the difference is the Harmonic series:

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} + \frac{1}{2^n}\right) - \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

and we know that the Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge! Thus, it cannot be true that $\sum_{n=1}^{\infty} \left(\frac{1}{n} + \frac{1}{2^n}\right)$ converges, in other words, it diverges!

3. The Integral Test

The Integral Test says the following.

> **The Integral Test.** Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence such that for every n it holds that $a_n \geq 0$ and $a_n = f(n)$ for some continuous, positive, and decreasing function f on the interval $[1, \infty)$. Then if the improper integral $\int_1^{\infty} f(x)\,dx$ converges (diverges) then the series $\sum_{n=1}^{\infty} a_n$ also converges (diverges).

It is important to note that you can apply the Integral Test only if you can show that $f(x)$ is a positive and decreasing function. It most cases, it will be clear that $f(x)$ is positive but to show that $f(x)$ is decreasing you can use the first derivative test which says that if the derivative $f'(x) < 0$ then $f(x)$ is decreasing.

Example 11: Determine if the sequence converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

Solution: The sequence $a_n = \frac{n}{n^2+1}$ is positive for all $n = 1, 2, 3 \ldots$. Consider the function $f(x) = \frac{x}{x^2+1}$. This function is decreasing for $x \geq 1$ because

$$f'(x) = \frac{1 - x^2}{(x^2+1)^2} < 0$$

when $x > 1$. It is clear that $f(x)$ is positive and continuous for $x \geq 1$. So, we can apply the Integral test:

$$\begin{aligned}\int_1^\infty \frac{x}{x^2+1}\,dx &= \lim_{t\to\infty}\int_1^t \frac{x}{x^2+1}\,dx \qquad \text{substitution } u=x^2+1\\ &= \lim_{t\to\infty}\frac{1}{2}\ln|x^2+1|\Big|_1^t\\ &= \lim_{t\to\infty}\left[\frac{1}{2}\ln|t^2+1|-\frac{1}{2}\ln|2|\right]\\ &= \infty\end{aligned}$$

because $\lim_{t\to\infty}\ln|t^2+1|=\infty$. Therefore, because the improper integral $\int_1^\infty \frac{x}{x^2+1}\,dx$ diverges then the series $\sum_{n=1}^\infty \frac{n}{n^2+1}$ also diverges.

Example 12: Determine if the sequence converges or diverges.

$$\sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$$

Solution: Notice that the index starts at $n=2$ and not at $n=1$. However, the Integral Test is valid on any interval $[N,\infty)$ where N is the starting point of the series. The sequence $a_n=\frac{1}{n(\ln n)^2}$ is positive for all $n=2,3,\ldots$. Consider the function $f(x)=\frac{1}{x(\ln x)^2}=(x(\ln x)^2)^{-1}$. It is clear that $f(x)$ is positive and continuous on the interval $[2,\infty)$. To see if it is decreasing compute its derivative:

$$f'(x)=(-1)(x(\ln x)^2)^{-2}(\ln(x)+2x\ln(x)\tfrac{1}{x})=-(x(\ln x)^2)^{-2}(\ln(x)+2\ln(x))=-\frac{3\ln(x)}{(x(\ln x)^2)^2}.$$

On the interval $[2,\infty)$, $f'(x)<0$, and so $f(x)$ is decreasing on the interval $[2,\infty)$. So, we can apply the Integral Test on the interval $[2,\infty)$:

$$\begin{aligned}\int_2^\infty \frac{1}{x(\ln x)^2}\,dx &= \lim_{t\to\infty}\int_2^t \frac{1}{x(\ln x)^2}\,dx \qquad \text{substitution } u=\ln x\\ &= \lim_{t\to\infty}-(\ln x)^{-1}\Big|_2^t\\ &= \lim_{t\to\infty}\left[-\frac{1}{\ln t}+\frac{1}{\ln(2)}\right]\\ &= \frac{1}{\ln(2)}\end{aligned}$$

Therefore, because the improper integral $\int_2^\infty \frac{1}{x(\ln x)^2}$ converges, the series $\sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$ also converges.

Example 13: Determine if the sequence converges or diverges.

$$\sum_{n=1}^\infty \frac{8\arctan(n)}{1+n^2}$$

Solution: The sequence $a_n = \frac{8\arctan(n)}{1+n^2}$ is positive for all $n = 1, 2, \ldots$. Consider the function $f(x) = \frac{8\arctan(x)}{1+x^2}$. Clearly, $f(x)$ is positive and continuous for $x \geq 1$. To see if it is decreasing, we compute:

$$f'(x) = \frac{1 - 2x\arctan(x)}{(1+x^2)^2}$$

The sign of $f'(x)$ depends only on the sign of the numerator $1 - 2x\arctan(x)$ because the denominator $(1+x^2)^2$ is clearly positive for all x. The function $2x\arctan(x)$ is an increasing function and when $x = 1$ we have $2(1)\arctan(1) = 2\frac{\pi}{4} = \frac{\pi}{2}$. Therefore the numerator at $x = 1$ is $1 - \frac{\pi}{2} < 0$. So, at $x = 1$, the numerator is negative. Since the term $2x\arctan(x)$ is increasing, we have $2\arctan(1) \leq 2x\arctan(x)$ for every $x \geq 1$, and therefore $-2\arctan(1) \geq -2x\arctan(x)$, and therefore

$$0 > 1 - 2\arctan(1) \geq 1 - 2x\arctan(x)$$

So, $f'(x) < 0$ for every $x \geq 1$, and therefore $f(x)$ is decreasing for $x \geq 1$. We can therefore apply the Integral Test:

$$\begin{aligned} \int_1^\infty \frac{8\arctan(x)}{1+x^2}\,dx &= \lim_{t\to\infty} \int_1^t \frac{8\arctan(x)}{1+x^2}\,dx \qquad \text{substitution } u = \arctan(x) \\ &= \lim_{t\to\infty} 8\frac{1}{2}(\arctan(x))^2\Big|_1^t \\ &= \lim_{t\to\infty} 4\left[(\arctan(t))^2 - (\arctan(1))^2\right] \\ &= 4\left[\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right] \end{aligned}$$

Therefore, because the improper integral $\int_1^\infty \frac{8\arctan(x)}{1+x^2}\,dx$ converges, the series $\sum_{n=1}^\infty \frac{8\arctan(n)}{1+n^2}$ also converges.