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A graph-theoretic classification for the controllability of the Laplacian leader-follower dynamics

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***Abstract*— In this paper, we revisit the controllability problem for the Laplacian based leader-follower dynamics with the aim of addressing some fundamental gaps within the existing literature. We introduce a notion of graph controllability classes for Laplacian based leader-follower control systems, namely, the classes of essentially controllable, completely uncontrollable, and conditionally controllable graphs. In addition to the topology of the underlying graph, our controllability classes rely on the richness of the set of control vectors. The particular focus in this paper is on the case where this set is chosen as the set of binary vectors, which captures the case when the control signal is broadcasted by the leader nodes. We first prove that the class of essentially controllable graphs is a strict subset of the class of asymmetric graphs. We provide a non-trivial class of completely uncontrollable asymmetric graphs, namely the class of large block graphs of Steiner triple systems. Several constructive examples demonstrate our results.**

I. INTRODUCTION

In recent years, there has been a surge of activity within the control theory community to understand how the network structure of multi-agent systems affects the fundamental properties of controllability and stabilizability. With regards to controllability, the growing body of literature has focused on the Laplacian based consensus dynamics, see for instance [1], [2], [3], [4], [5], [6], and references therein. Specifically, starting with a Laplacian consensus algorithm, a subset of the agents are classified as *leaders* and act as control agents that can change the dynamics of the network. The remaining agents, called the *followers*, are indirectly controlled by the leaders via the connectivity of the network. Most of the effort in the current literature has focused on obtaining graph theoretic conditions under which such systems are uncontrollable. For example, in [2] it is shown that if leader nodes are chosen that preserve a non-trivial graph symmetry then the resulting system is uncontrollable.

In this paper, we introduce graph controllability classes for the Laplacian leader-follower dynamics over undirected graphs, namely, essentially controllable, completely uncontrollable, and conditionally controllable graphs. In addition to the topology of the underlying graph, our graph controllability classes rely on the richness of the set of control vectors. Essentially controllable graphs are controllable for any choice of non-trivial control vectors, completely uncontrollable graphs are uncontrollable for any choice of the control vector, and finally, conditionally controllable graphs are controllable for a strict subset of the control vectors. The particular focus in this paper is on the case where the control set is the binary vectors. The motivation for this choice is to capture the situation where the control signal is broadcasted by the leader nodes. Moreover, it includes as a special case the scenario considered in [2]. We prove that essentially controllable graphs are necessarily asymmetric. Notably, we show that the example of the 6-graph provided in the literature [2] to demonstrate that asymmetry is not necessary for uncontrollability, is in fact essentially controllable; thus this is not an appropriate example of such phenomenon.

We next focus on the class of completely uncontrollable graphs. We provide an explicit class of graphs, namely, the class of block graphs of Steiner triple systems, that are asymmetric yet completely uncontrollable. This result indicates that characterizing graph uncontrollability via graph symmetries targets a narrow class. Given that having a repeated Laplacian eigenvalue results in complete uncontrollability, we prove that completely uncontrollable graphs on four and five vertices are completely uncontrollable if and only if they have a repeated eigenvalue. Numerical evidence suggests that the same result holds for graphs of orders six and seven. However, we provide explicit examples of graphs on eight and nine vertices that are completely uncontrollable and yet have simple eigenvalues. Throughout the paper, several examples demonstrate the results.

II. PRELIMINARIES

In this section, we establish notation and basic notions from graph theory, and state a result on the controllability of linear systems when the system matrix is diagonalizable.

A. Graph theory

Our notation from graph theory is standard, see for instantce [7], [8]. By a *graph* we mean a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consisting of a finite *vertex* set $\mathcal{V}$ and an *edge* set $\mathcal{E} \subseteq [\mathcal{V}]^2 := \{\{v, w\} \mid v, w \in \mathcal{V}\}$. The *order* of the graph $\mathcal{G}$ is the cardinality of its vertex set $\mathcal{V}$. The *neighbors* of $v \in \mathcal{V}$ is the set $\mathcal{N}_v := \{w \in \mathcal{V} \mid \{v, w\} \in \mathcal{E}\}$ and the *degree* of v, denoted d_v, is cardinality of $\mathcal{N}_v$, that is, $d_v := |\mathcal{N}_v|$. A graph $\mathcal{G}$ is *connected* if there is a path between any pair of vertices, that is, given a pair of vertices v and w there is a sequence of distinct vertices $(v_0, v_1, \ldots, v_k)$ such that $\{v_{i-1}, v_i\} \in \mathcal{E}$ and $v_0 = v$ and $v_k = w$.

Henceforth, without loss of generality, we let $\mathcal{V} = \{1, \ldots, n\}$, where n is the order of $\mathcal{G}$. The *adjacency matrix* of $\mathcal{G}$ is the $n \times n$ matrix $\mathbf{A}$ defined as $\mathbf{A}_{ij} = 1$ if $\{i, j\} \in \mathcal{E}$

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and $\mathbf{A}_{ij} = 0$ otherwise, where $\mathbf{A}_{ij}$ denotes the entry of $\mathbf{A}$ in the ith row and jth column.

We denote by $\mathbf{D}$ the degree matrix of $\mathcal{G}$, i.e., the diagonal matrix whose ith diagonal entry is d_i. The Laplacian matrix of $\mathcal{G}$ is given by

$$\mathbf{L} = \mathbf{D} - \mathbf{A}.$$

The Laplacian matrix $\mathbf{L}$ is symmetric and positive semi-definite, and thus the eigenvalues of $\mathbf{L}$ can be ordered $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. The ones vector $\mathbf{1}_n := [1\ 1\ \cdots\ 1]^T$ is an eigenvector of $\mathbf{L}$ with eigenvalue $\lambda_1 = 0$, and if $\mathcal{G}$ is connected then $\lambda_1 = 0$ is a simple eigenvalue of $\mathbf{L}$. We assume throughout that $\mathcal{G}$ is connected so that $0 < \lambda_2$.

A mapping $\varphi : \mathcal{V} \to \mathcal{V}$ is an *automorphism* of $\mathcal{G}$ if it is a bijection and $\{i, j\} \in \mathcal{E}$ implies that $\{\varphi(i), \varphi(j)\} \in \mathcal{E}$. An automorphism φ of $\mathcal{G}$ induces a linear transformation on $\mathbb{R}^n$, denoted by $\mathbf{P}_\varphi$ or just $\mathbf{P}$ when φ is understood, whose matrix representation in the standard basis is a permutation matrix, i.e., as a linear mapping φ acts as a permutation on the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$. It is well known and straightforward to show that φ is an automorphism of $\mathcal{G}$ if and only if $\mathbf{PA} = \mathbf{AP}$. Moreover, an automorphism $\mathbf{P}$ preserves degree of vertices and therefore $d_i = d_{\varphi(i)}$ for every $i \in \{1, 2, \ldots, n\}$, in other words $\mathbf{PD} = \mathbf{DP}$, and consequently $\mathbf{PL} = \mathbf{LP}$.

A graph is called *k-regular* if all its vertices have degree $k \in \mathbb{N}$. A k-regular graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called *strongly regular*, denoted by $\mathrm{SGR}(n, k, \lambda, \mu)$, if there exists $\lambda, \mu \in \mathbb{N}$ such that

i) $|\mathcal{N}_v \cap \mathcal{N}_u| = \lambda$, for every $v \in \mathcal{V}$ and every $u \in \mathcal{N}_v$;
ii) $|\mathcal{N}_v \cap \mathcal{N}_u| = \mu$, for every $v \in \mathcal{V}$ and every $u \notin \mathcal{N}_v$.

It is known that strongly regular graphs have exactly three eigenvalues [9]. A subclass of strongly regular graphs, the so-called *block graph* of a *Steiner triple system*, plays an important role in one of our main results.

Definition 2.1: **(Steiner triple systems):** A (t, k, n)-*Steiner triple system* of order n, denoted by $\mathrm{STS}(t, k, n)$, is a set S of n elements together with a set of k-element subsets of S (called *blocks*) such that any t elements of the set S is contained exactly in one block.

A Steiner triple system of order $n > 1$ exists if and only if $n = 1$ or $3 (\mathrm{mod}\ 6)$ [10]. The *block graph* of a Steiner triple system $\mathrm{STS}(t, k, n)$ is the graph $\mathcal{G}_{\mathrm{STS}}$ with the k blocks as vertices, and where two blocks are adjacent when they have nonempty intersection. By definition, such a graph is strongly regular. The first nontrivial Steiner triple system is the Fano plane, which has 7 blocks, each containing 3 points, and every pair of points belongs to a unique line.

B. Linear Controllability and Diagonalizability

Given a matrix $\mathbf{F} \in \mathbb{R}^{n \times n}$ and vector $\mathbf{b} \in \mathbb{R}^n$, we denote by $\langle \mathbf{F}; \mathbf{b} \rangle$ the smallest $\mathbf{F}$-invariant subspace containing $\mathbf{b}$. It is well-known that $\langle \mathbf{F}; \mathbf{b} \rangle = \mathrm{span}\{\mathbf{F}^k \mathbf{b} \mid k \in \mathbb{N}_0\}$, and that if $\dim(\langle \mathbf{F}; \mathbf{b} \rangle) = k + 1$ then $\{\mathbf{b}, \mathbf{Fb}, \ldots, \mathbf{F}^k \mathbf{b}\}$ is a basis for $\langle \mathbf{F}; \mathbf{b} \rangle$. The pair $(\mathbf{F}, \mathbf{b})$ is called controllable if $\dim(\langle \mathbf{F}; \mathbf{b} \rangle) = n$.

The following result characterizes the controllability of single-input linear systems $(\mathbf{F}, \mathbf{b})$ when $\mathbf{F}$ is diagonalizable.

Proposition 2.1: **(Controllability and eigenvalue multiplicity):** Let $\mathbf{F} \in \mathbb{R}^{n \times n}$ and suppose that $\mathbf{F}$ is diagonalizable. The following hold:

(i) For any open set $\mathcal{B} \subset \mathbb{R}^n$, the pair $(\mathbf{F}, \mathbf{b})$ is uncontrollable for every $\mathbf{b} \in \mathcal{B}$ if and only if $\mathbf{F}$ has a repeated eigenvalue.
(ii) Suppose that $\mathbf{F}$ has distinct eigenvalues and let $\mathbf{U}$ be a matrix whose columns are linearly independent eigenvectors of $\mathbf{F}$. If $\mathbf{b} \in \mathbb{R}^n$ then the dimension of $\langle \mathbf{F}; \mathbf{b} \rangle$ is equal to the number of nonzero components of $\mathbf{v} = \mathbf{U}^{-1}\mathbf{b}$. In particular, $(\mathbf{F}, \mathbf{b})$ is controllable if and only if no component of $\mathbf{v}$ is zero.

The proof of (i) follows from the properties of the determinant and for the proof of (ii) see for instance [1].

III. Problem Statement and Preliminary Results

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with vertex set $\mathcal{V} = \{1, 2, \ldots, n\}$. The Laplacian dynamics on $\mathcal{G}$ is the linear system

$$\dot{\mathbf{x}}(t) = -\mathbf{L}\mathbf{x}(t),$$

where $\mathbf{x} \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Suppose that a subset of the vertices $\widetilde{\mathcal{V}} \subset \mathcal{V}$ are actuated by a single control $u : [0, \infty) \to \mathbb{R}$ and consider the resulting single-input linear control system. Explicitly, let $\mathbf{b} = [b_1\ b_2\ \cdots\ b_n]^T \in \{0, 1\}^n$ be the binary vector such that $\widetilde{\mathcal{V}} = \mathcal{V}_\mathbf{b} := \{i \in \mathcal{V} \mid b_i = 1\}$, and consider the single-input linear control system

$$\dot{\mathbf{x}}(t) = -\mathbf{L}\mathbf{x}(t) + \mathbf{b}u(t). \tag{1}$$

The vertices $\mathcal{V}_\mathbf{b}$ are seen as control or *leader* nodes and influence the remaining *follower* nodes $\mathcal{V} \backslash \mathcal{V}_\mathbf{b}$ through the control signal $u(\cdot)$ and the connectivity of the network. The motivation behind the set of binary control vectors is that it captures the scenario of when an external agent connected to the nodes $\mathcal{V}_\mathbf{b}$ is unable to distinguish between its followers. Hence, all the followers receive the same control input from the leader, i.e., the control signal is broadcasted. The reason for choosing the Laplacian dynamics (1) is that it serves as a benchmark problem for studying distributed control systems; nevertheless, the ideas that will be developed in this paper can be extended to other classes of sparse positive systems. We also note that the approach taken in [2] is a special case of the problem we consider here (see Remark 3.1).

From a controls design perspective, it is desirable to select the leader nodes so that the pair $(\mathbf{L}, \mathbf{b})$ is controllable. For example, as a particular case of Proposition 2.1(ii), the choice of $\mathbf{b} = \mathbf{1}_n$ results in a controllable pair $(\mathbf{L}, \mathbf{b})$ if and only if $n = 1$, since $\mathbf{1}_n$ is orthogonal to every other eigenvector of $\mathbf{L}$ (and of course $\mathbf{L}\mathbf{1}_n = \mathbf{0}_n$). In fact, an immediate consequence of Proposition 2.1 for the Laplacian dynamics is the following result.

Corollary 3.1 ([1]): **(Necessary and sufficient condition for controllability of Laplacian dynamics):** Consider the controlled Laplacian dynamics (1) with $\mathbf{b} \in \mathbb{R}^n$ and assume that $\mathbf{L}$ has no repeated eigenvalues. Then the pair $(\mathbf{L}, \mathbf{b})$

is controllable if and only if $\mathbf{b}$ is not orthogonal to any eigenvector of $\mathbf{L}$.

Although Corollary 3.1 gives a characterization of the control vectors $\mathbf{b}$ that result in controllability, the problem that we consider is in obtaining graph theoretic characterizations of such $\mathbf{b}$'s. One such characterization was obtained in [2] (see also [11, Lemma 1.1]) in terms of the automorphism group of the graph $\mathcal{G}$. To this end, following [2], we say that $\mathbf{b} \in \{0,1\}^n$ is *leader symmetric* if there exists a non-trivial automorphism $\varphi : \mathcal{V} \to \mathcal{V}$ of $\mathcal{G}$ that leaves the set $\mathcal{V}_{\mathbf{b}}$ invariant, i.e., $\varphi(\mathcal{V}_{\mathbf{b}}) = \mathcal{V}_{\mathbf{b}}$. The following result links leader symmetry and uncontrollability.

Proposition 3.1 ([2]): **(Leader symmetry and uncontrollability):** Consider the controlled Laplacian dynamics (1) with $\mathbf{b} \in \{0,1\}^n$. If $\mathbf{b}$ is leader symmetric then $(\mathbf{L}, \mathbf{b})$ is uncontrollable.

As shown in [2, Proposition 5.9], leader symmetry is not a necessary condition for uncontrollability. Figure 1(a) depicts the graph on $n = 6$ vertices that is used in [2] to show this fact. Unfortunately, this example is not illuminating because the leader nodes are chosen so that $\mathbf{b} = \mathbf{1}_n$, i.e., every node is actuated. By the remark made above, unless $n = 1$, this choice always results in uncontrollability, regardless of the graph topology. Furthermore, an interesting fact is true for the graph of Figure 1(a); the only $\mathbf{b}$'s for which this graph is uncontrollable are the trivial cases $\mathbf{b} = \mathbf{0}_n$ or $\mathbf{b} = \mathbf{1}_n$. We call such graphs *essentially controllable* and we will return to them in Section IV. Let us give an example of an asymmetric graph having many non-trivial $\mathbf{b}$'s resulting in uncontrollability.

Example 3.1: **(Leader symmetry is not necessary for uncontrollability):** Consider the graph in Figure 1(b) with $n = 6$ vertices. This graph is asymmetric, see for instance [12]. There are 14 of the $2^n - 2 = 62$ non-trivial choices of $\mathbf{b}$ that makes (1) uncontrollable, namely:

$$\begin{aligned}
\mathbf{b}_1 &= (1,1,1,0,0,0) & \mathbf{b}_2 &= (0,0,0,1,1,1) \\
\mathbf{b}_3 &= (1,1,0,1,0,0) & \mathbf{b}_4 &= (0,0,1,0,1,1) \\
\mathbf{b}_5 &= (0,1,1,1,0,0) & \mathbf{b}_6 &= (1,0,0,0,1,1) \\
\mathbf{b}_7 &= (0,1,0,0,1,0) & \mathbf{b}_8 &= (1,0,1,1,0,1) \\
\mathbf{b}_9 &= (1,0,1,0,1,0) & \mathbf{b}_{10} &= (0,1,0,1,0,1) \\
\mathbf{b}_{11} &= (1,0,0,1,1,0) & \mathbf{b}_{12} &= (0,1,1,0,0,1) \\
\mathbf{b}_{13} &= (0,0,1,1,1,0) & \mathbf{b}_{14} &= (1,1,0,0,0,1)
\end{aligned}$$

The control vectors $\mathbf{b}_7$ and $\mathbf{b}_8$ result in a 2-dimensional controllable subspace, while the other control vectors all result in a 5-dimensional controllable subspace. □

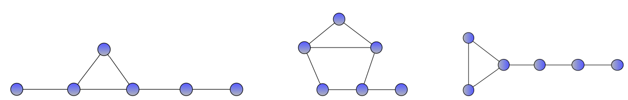


Fig. 1. (a) The example of [2], (b) an asymmetric graph on $n = 6$ vertices having 14 binary vectors $\mathbf{b}$ resulting in uncontrollable Laplacian dynamics, and (c) a graph for which any binary vector $\mathbf{b}$ results in uncontrollability.

The motivating example above, and more importantly that asymmetry is typical in finite graphs [12], suggests that leader symmetry, although important, is a coarse topological graph controllability obstruction. To better understand how the topology of the graph affects controllability, in the next section we introduce *graph controllability classes*, present some preliminary characterizations of their properties, and present some examples.

Remark 3.1: **(Comparison with [2]):** Let us describe the approach taken in [2] and how it relates to ours. We note that our approach is also adopted in [5], [6]. In [2], one begins with a Laplacian based dynamics $\dot{\mathbf{x}} = -\mathbf{L}\mathbf{x}$, selects a leader node, say $i \in \{1, 2, \ldots, n\}$, and considers the reduced system of followers actuated by node i. Explicitly, following [2] let $\mathbf{L}_f \in \mathbb{R}^{(n-1)\times(n-1)}$ be the matrix obtained by deleting the ith row and ith column of $\mathbf{L}$, and let $\mathbf{b}_f \in \mathbb{R}^{n-1}$ be column vector obtained by removing the ith entry of the ith column of $\mathbf{L}$. The reduced system of followers considered in [2] is $\dot{\mathbf{z}} = -\mathbf{L}_f\mathbf{z} - \mathbf{b}_f u$. It is not hard to show that the system $(\mathbf{L}_f, \mathbf{b}_f)$ is controllable if and only if $(\mathbf{L}, \mathbf{e}_i)$ is controllable. Indeed, the dynamic extension

$$\begin{aligned}
\dot{\mathbf{z}} &= -\mathbf{L}_f\mathbf{z} - \mathbf{b}_f\xi, \\
\dot{\xi} &= v,
\end{aligned}$$

is controllable if and only if $(\mathbf{L}_f, \mathbf{b}_f)$ is controllable. Letting $v = -\mathbf{b}_f^T\mathbf{z} - d_i\xi + u$, we see that the dynamic extension is feedback equivalent to $(\mathbf{L}, \mathbf{e}_i)$. Hence, in relation to the problem we consider in this paper, the approach in [2] is concerned with the controllability of $(\mathbf{L}, \mathbf{b})$ in the restricted case that $\mathbf{b} \in \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\} \subset \{0,1\}^n$. We note that the graph in Example 3.1 is such that $(\mathbf{L}, \mathbf{e}_i)$ is controllable for every $\mathbf{e}_i \in \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, yet as shown in the example, fails to be controllable for some $\mathbf{b} \in \{0,1\}^n$.

IV. Graph Controllability Classes

In this section, we introduce controllability classes for the controlled Laplacian dynamics (1). We assume that the control vectors may be chosen from a set $\mathcal{B} \subset \mathbb{R}^n$, and thus controllability, or lack thereof, is with respect to the set $\mathcal{B}$. This naturally results in not just controllable and uncontrollable systems but also on partially or conditionally controllable systems.

Definition 4.1: **(Graph controllability classes):** Let $\mathcal{G}$ be a connected graph with Laplacian matrix $\mathbf{L}$ and let $\mathcal{B} \subset \mathbb{R}^n$ be a non-empty set. Then $\mathcal{G}$ is called

(i) *essentially controllable* on $\mathcal{B}$ if $(\mathbf{L}, \mathbf{b})$ is controllable for every $\mathbf{b} \in \mathcal{B}\setminus \ker(\mathbf{L})$;
(ii) *completely uncontrollable* on $\mathcal{B}$ if $(\mathbf{L}, \mathbf{b})$ is uncontrollable for every $\mathbf{b} \in \mathcal{B}$;
(iii) *conditionally controllable* on $\mathcal{B}$ if it is neither essentially controllable nor completely uncontrollable on $\mathcal{B}$.

In this paper we are concerned with controllability classes on the control set $\mathcal{B} = \{0,1\}^n$. Hence, when not explicitly stated, we simply call a graph $\mathcal{G}$ essentially controllable (conditionally controllable, or completely uncontrollable) if

$\mathcal{G}$ is essentially controllable (conditionally controllable, or completely controllable) on $\{0,1\}^n$.

Example 4.1: **(Graph controllability classes):** According to Definition 4.1, the graph in Figure1(a) is essentially controllable, while the graph in Figure1(b) is conditionally controllable. Finally, it is easy to verify that the graph in Figure1(c) is completely uncontrollable on $\{0,1\}^6$. •

V. Graph-theoretic characterization of controllability classes

In this section, we present preliminary results on characterizing the class of essentially controllable and completely uncontrollable graphs. Before we state our main results, we first provide a useful property of controllability under binary control vectors.

A. Invariance of Controllability Under Binary Complements

The reader may have noticed that the binary control vectors listed in Example 3.1 come in complementary pairs (compare each vector on the left column with the corresponding vector on the right). To be more precise, given $\mathbf{b} \in \{0,1\}^n$ we let

$$\overline{\mathbf{b}} = \mathbf{1}_n - \mathbf{b}$$

be the *complement* of $\mathbf{b}$. With this notation we have the following result.

Proposition 5.1: **(Controllability and binary complements):** Let $n \geq 2$ and consider the controlled Laplacian dynamics (1) with $\mathbf{b} \in \{0,1\}^n$. Then the pair $(\mathbf{L}, \mathbf{b})$ is controllable if and only if the pair $(\mathbf{L}, \overline{\mathbf{b}})$ is controllable. In fact, the controllability matrices of $(\mathbf{L}, \mathbf{b})$ and $(\mathbf{L}, \overline{\mathbf{b}})$ have the same rank provided $\mathbf{b} \notin \{\mathbf{1}_n, \mathbf{0}_n\}$.

Proof: We give only a sketch of the proof. Let $\mathbf{b} \in \{0,1\}^n \backslash \{\mathbf{1}_n, \mathbf{0}_n\}$, let $\mathbf{v} = \mathbf{U}^T\mathbf{b} = [v_1 \ \cdots \ v_n]^T$, and let $\bar{\mathbf{v}} = \mathbf{U}^T\overline{\mathbf{b}} = [\bar{v}_1 \ \cdots \ \bar{v}_n]^T$. The claim follows by noticing that

$$\bar{\mathbf{v}} = \tfrac{n}{\sqrt{n}}\mathbf{e}_1 - \mathbf{v}. \tag{2}$$

and using Proposition 2.1(ii). ■

For computational purposes, it is worth mentioning the following immediate consequence of the previous result.

Corollary 5.1: **(Uncontrollable subspace has even cardinality):** If $n \geq 2$ then the cardinality of the set $\{\mathbf{b} \in \{0,1\}^n \mid (\mathbf{L}, \mathbf{b}) \text{ is uncontrollable}\}$ is even.

B. Asymmetry of essentially controllable graphs

In this section, we give a necessary condition for essential controllability. The condition depends on the following auxiliary result.

Lemma 5.1: **(Order of the non-identity automorphisms [7]):** If all of the eigenvalues of $\mathbf{L}$ are simple then every non-identity automorphism of $\mathcal{G}$ has order two.

Proof: The proof of the claim when $\mathbf{L}$ is replaced by the adjacency matrix $\mathbf{A}$ is given in [7, Theorem 15.4]. However, the proof for the case of $\mathbf{L}$ is identical because if $\mathbf{P}$ is an automorphism of $\mathcal{G}$ then $\mathbf{P}$ commutes with both the adjacency matrix $\mathbf{A}$ and the degree matrix $\mathbf{D}$, and therefore $\mathbf{P}$ also commutes with $\mathbf{L}$. ■

Proposition 5.2: **(Essentially controllable graphs are asymmetric):** An essentially controllable graph on $\{0,1\}^n$ is asymmetric.

Proof: Let $\mathcal{G}$ be an essentially controllable graph on $\{0,1\}^n$. Then necessarily $\mathbf{L}$ must have distinct eigenvalues and therefore, by Lemma 5.1, every non-identity automorphism of $\mathcal{G}$ has order two. Assume by contradiction that $\mathcal{G}$ has a non-trivial automorphism group and let $\mathbf{P}$ be a permutation matrix representing a non-identity automorphism of $\mathcal{G}$. Since $\mathbf{P}$ has order two, i.e., $\mathbf{P}^2 = \mathbf{I}_{n\times n}$, there exists two distinct standard basis vectors $\mathbf{e}_i$ and $\mathbf{e}_j$ such that $\mathbf{P}\mathbf{e}_i = \mathbf{e}_j$ and $\mathbf{P}\mathbf{e}_j = \mathbf{e}_i$. Put $\mathbf{b} = \mathbf{e}_i + \mathbf{e}_j$. Then $\mathbf{b}$ is clearly invariant under $\mathbf{P}$, i.e., $\mathbf{P}\mathbf{b} = \mathbf{b}$. Thus, $\mathbf{b}$ is leader symmetric and therefore, by Proposition 3.1, $(\mathbf{L}, \mathbf{b})$ is uncontrollable, which is a contradiction. This completes the proof. ■

According to Proposition 5.2, and since any asymmetric graph has at least six vertices [12], any essentially controllable graph also has at least six vertices. The condition given in Proposition 5.2 is, however, clearly only necessary; the graph in Figure 1(b) is an example of an asymmetric graph with six nodes which is not essentially controllable.

Example 5.1: **(Essentially controllable graphs on $\{0,1\}^n$):** Exactly four of the eight asymmetric graphs on six vertices are essentially controllable; these graphs are shown in Figure 2.

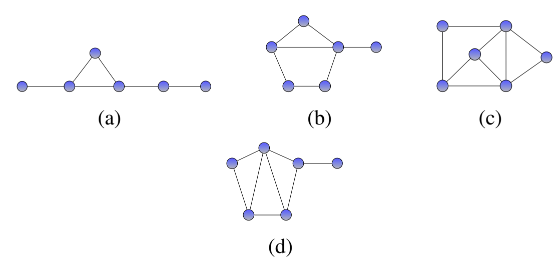


Fig. 2. All essentially controllable graph with six vertices.

The class of essentially controllable graphs are interesting for various reasons. First, this class is important from a design perspective because, except for the trivial inputs of $\mathbf{0}_n$ and $\mathbf{1}_n$, controllability is independent of the subset of nodes that receive the control inputs. This is useful when this is unknown a priori such as in the context where the control inputs are broadcasted. Another important fact about essentially controllable graphs is that the so-called minimal controllability problem is solvable [13] for this class of graphs. Following [13], let $\mathcal{B} \subset \mathbb{R}^n$ and consider the dynamics (1) for fixed $\mathbf{b} \in \mathcal{B}$. We say that (1) is *minimally controllable* if $\mathbf{b}$ has the fewest number of nonzero entries among all vectors $\tilde{\mathbf{b}} \in \mathcal{B}$ such that $(\mathbf{L}, \tilde{\mathbf{b}})$ controllable. It is shown in [13] that it is in general intractable to even approximate the number of zeros in the vector $\mathbf{b}$ that leads to minimal controllability. Nevertheless, given that the class of essentially controllable graphs are controllable using any non-trivial vector in $\{0,1\}^n$, the minimal controllability problem is solvable for (1) on all essentially controllable graphs, and the sparsest $\mathbf{b} \in \{0,1\}^n$ has $(n-1)$ nonzero entries.

We are not aware of any algorithm producing essentially controllable graphs. Given that these graphs constitute a strict subset of asymmetric graphs, and that it is NP-hard to verify if a graph has non-trivial automorphisms [14], we are not aware if the problem of generating essentially controllable graphs of order n is computationally feasible. Another interesting problem is to investigate how the number of essentially controllable graphs grows with respect to the number of asymmetric graphs.

C. The class of completely uncontrollable graphs

In this section, we study the class of completely uncontrollable graphs. Our first result illustrates that the intersection of the class of completely uncontrollable graphs and asymmetric graphs is nonempty and large.

1) Uncontrollability and asymmetric topologies: Here, we provide an explicit class of arbitrarily large graphs that are asymmetric and completely uncontrollable. This demonstrates that having a trivial graph automorphism group does not eliminate the possibility of complete uncontrollability. To the best of our knowledge, this important fact is unknown in the literature on network controllability primarily because most existing results deal with the characterization of uncontrollability via graph symmetries.

Theorem 5.1: **(A class of completely uncontrollable asymmetric graphs):** For any $N \geq 1$ there exists a connected and asymmetric graph of order $n \geq N$ that is completely uncontrollable.

Proof: The class of block graphs of Steiner triple systems, see Definition 2.1, are almost always asymmetric [15, Theorem 1]. Since any block graph of a Steiner triple system is strongly regular, its adjacency matrix only has three distinct eigenvalues and hence, by regularity, its Laplacian also has only three distinct eigenvalues. The result is then a direct consequence of Proposition 2.1 (i). ■

The completely uncontrollable graphs in Theorem 5.1 have a repeated eigenvalue. In the following section we investigate whether this is a general necessary condition for complete uncontrollability.

2) An algebro-geometric characterization of completely uncontrollable graphs: If $\mathbf{F} \in \mathbb{R}^{n\times n}$ is diagonalizable then by Proposition 2.1(i), for any open subset $\mathcal{B} \subset \mathbb{R}^n$ the pair $(\mathbf{F}, \mathbf{b})$ is uncontrollable for every $\mathbf{b} \in \mathcal{B}$ if and only if $\mathbf{F}$ has a repeated eigenvalue. When $\mathcal{B}$ is replaced by a discrete set, such as $\mathcal{B} = \{0,1\}^n$, the condition of a repeated eigenvalue is no longer necessary for uncontrollability. For example, the symmetric matrix

$$\mathbf{F} = \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 5 & -3 \\ -1 & -1 & -3 & 5 \end{bmatrix}$$

has distinct eigenvalues $\lambda_1 = 0, \lambda_2 = 2, \lambda_3 = 4, \lambda_4 = 8$, and it is readily verified that $(\mathbf{F}, \mathbf{b})$ is uncontrollable for every $\mathbf{b} \in \{0,1\}^4$. Of course, $\mathbf{F}$ is not the Laplacian matrix of any (undirected) graph. The problem of complete uncontrollability for the Laplacian leader-follower dynamics can be casted as a geometric problem. To do so, we need the following definition.

Definition 5.1: **($\mathcal{B}$-anniliators):** Let $\mathcal{B} \subset \mathbb{R}^n$ and let $\gamma = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \subset \mathbb{R}^n$ be linearly independent. We say that γ is a **$\mathcal{B}$-anniliator** or that it **annihilates** $\mathcal{B}$ if for each $\mathbf{b} \in \mathcal{B}$ there exists $\mathbf{u}_j \in \gamma$ that is orthogonal to $\mathbf{b}$, that is, $\mathbf{u}_j^T\mathbf{b} = 0$. Consider now the following problem.

Problem 5.1: **(Algebro-geometric formulation of complete uncontrollability on $\{0,1\}^n$):** Does there exist an orthonormal basis $\gamma = \{\mathbf{u}_1, ..., \mathbf{u}_n\}$ for $\mathbb{R}^n$, with $\mathbf{u}_1 = \frac{1}{\sqrt{n}}\mathbf{1}_n$, such that

UC1. γ is a $\{0,1\}^n$-annihilator,

UC2. γ is a set of eigenvectors of the Laplacian matrix $\mathbf{L}$ of a connected graph, and

UC3. $\mathbf{L}$ has **no repeated eigenvalues**?

In dimensions $n = 2$ and $n = 3$, it is not hard to show that no basis γ exists that satisfies even **UC1.** However, for $n = 4$ and $n = 5$, we have proved that completely uncontrollable graphs have a repeated eigenvalue.

Proposition 5.3: **(Completely uncontrollable graphs with four and five vertices):** All connected and completely uncontrollable graphs on $\{0,1\}^4$ and $\{0,1\}^5$ have a repeated eigenvalue.

Our numerical computations show that for $n = 6$ and $n = 7$, complete uncontrollability is fully characterized by having a repeated eigenvalue. Surprisingly, this is no longer the case for $n \geq 8$. In Figure 3, we display two such graphs on $n = 8$ vertices and one for $n = 9$ vertices. Needless to say, the class

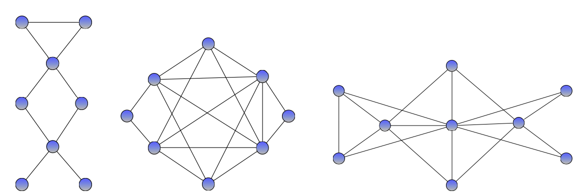


Fig. 3. (a) and (b) show two completely uncontrollable graphs with $n = 8$ vertices and (c) shows a completely uncontrollable graphs with $n = 9$ vertices, all with distinct eigenvalues.

of completely uncontrollable graphs with distinct eigenvalues form a very special class of graphs and have the potential to shed light on new necessary conditions for controllability and will be pursued in a future paper.

Interestingly, the proof of Proposition 5.3 identifies a set of three vectors that *alone* are $\{0,1\}^n$-annihilators. For $n \geq 4$, define

$$\begin{aligned} \mathbf{v}_1 &= \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}^T, \\ \mathbf{v}_2 &= \begin{bmatrix} 0 & 0 & 1 & -1 & 0 & \cdots & 0 \end{bmatrix}^T, \\ \mathbf{v}_3 &= \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & \cdots & 0 \end{bmatrix}^T, \end{aligned} \tag{3}$$

We have the following.

Proposition 5.4: **(A set of $\{0,1\}^n$-annihilator vectors):** Let $n \geq 4$ and consider the set $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are given by (3). Then β is a $\{0,1\}^n$-annihilator.

Proof: Any vector $\mathbf{b} \in \{0,1\}^n$ having a zero in components 1 through 4 is clearly orthogonal to $\mathbf{v}_1$ (and $\mathbf{v}_2$, and $\mathbf{v}_3$). Therefore, we need only consider the $\{0,1\}$ vectors having possibly nonzero entries in components $1, 2, 3$, and/or 4. There are $\sum_{k=1}^{4} \binom{4}{k} = 15$ possible cases to consider, and the details are left to the reader. ∎

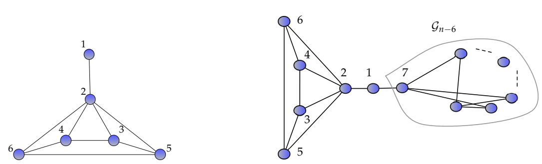
Using Proposition 5.4 we can identify a class of non-regular completely uncontrollable graphs.

Theorem 5.2: **(Large uncontrollable graphs):** For each $n \geq 6$, the set of graphs of order n that are not regular and completely uncontrollable is non-empty.

Proof: We give a sketch of the proof. For $n = 6$, consider the graph in Figure 4(a) and denote its Laplacian matrix by $\mathbf{L}_6$, which has linearly independent eigenvectors

$$\begin{aligned}
\mathbf{u}_1 &= \tfrac{1}{\sqrt{6}}\mathbf{1}_6^T, \\
\mathbf{u}_2 &= \tfrac{1}{\sqrt{30}}\begin{bmatrix}5 & -1 & -1 & -1 & -1 & -1\end{bmatrix}^T, \\
\mathbf{u}_3 &= \tfrac{1}{\sqrt{2}}\begin{bmatrix}0 & 0 & -1 & 0 & 0 & 1\end{bmatrix}^T, \\
\mathbf{u}_4 &= \tfrac{1}{\sqrt{2}}\begin{bmatrix}0 & 0 & 0 & 1 & -1 & 0\end{bmatrix}^T, \\
\mathbf{u}_5 &= \tfrac{1}{2}\begin{bmatrix}0 & 0 & 1 & -1 & -1 & 1\end{bmatrix}^T, \\
\mathbf{u}_6 &= \tfrac{1}{\sqrt{20}}\begin{bmatrix}-4 & 0 & 1 & 1 & 1 & 1\end{bmatrix}^T.
\end{aligned}$$

After a permutation of the indices, we can apply Proposition 5.4 to the set $\{\mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\}$ to conclude that it is a $\{0,1\}^6$-annihilator. Now let $n \geq 6$ and extend the graph in Figure 4(a) to the graph $\mathcal{G}$ shown in Figure 4(b), where $\mathcal{G}_{n-6}$ is any connected graph on $n-6$ vertices. By construction,



(a) (b)

Fig. 4. A $\{0,1\}^n$-annihilator graph with six verices (a), and its extension to a $\{0,1\}^n$-annihilator graph of any size (b).

the Laplacian of $\mathcal{G}$ can be decomposed as

$$\mathbf{L} = \begin{bmatrix}\mathbf{L}_6 & \mathbf{E} \\ \mathbf{E}^T & \mathbf{L}_{n-6}\end{bmatrix},$$

where $\mathbf{L}_{n-6}$ denotes the Laplacian of the graph $\mathcal{G}_{n-6}$ and $\mathbf{E} \in \mathbb{R}^{6\times(n-6)}$ is the matrix

$$\mathbf{E} = \begin{bmatrix}-\mathbf{e}_1 & \mathbf{0}_n & \cdots & \mathbf{0}_n\end{bmatrix}$$

From the above decomposition of $\mathbf{L}$, and noting that the first entries of $\mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$ are zero, it is not hard to show that $\mathbf{u}_3, \mathbf{u}_4$ and $\mathbf{u}_5$ can be lifted to eigenvectors of $\mathbf{L}$. It is clear that the lifted eigenvectors $\left[\begin{smallmatrix}\mathbf{u}_j \\ \mathbf{0}_{n-6}\end{smallmatrix}\right] \in \mathbb{R}^n$, for $j \in \{3,4,5\}$, form a set of $\{0,1\}^n$ annihilators. This ends the proof. ∎

The next result, whose proof is omitted due to space limitations, states that any graph containing the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in (3) as eigenvectors will have a repeated eigenvalue.

Proposition 5.5: **($\{0,1\}^n$-annihilator graphs and repeated eigenvalues):** Let $\mathcal{G}$ be a graph on $n \geq 4$ vertices. If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ given by (3) are eigenvectors of $\mathcal{G}$ then $\mathcal{G}$ has a repeated eigenvalue.

Conclusion and Future Work

We have proposed a classification for controllability of the Laplacian leader-follower dynamics by introducing the class of essentially controllable, completely uncontrollable, and conditionally controllable graphs. We have presented preliminary results for the characterization of completely uncontrollable and essentially controllable classes. In particular, we have showed that all essentially controllable graphs are asymmetric and have provided large classes of asymmetric completely uncontrollable graphs. We have also proved the necessity of having repeated eigenvalues for the Laplacian matrix for complete uncontrollability for graphs of low order, and a class of large graphs.

A complete characterization of complete uncontrollability, investigating the existence of a polynomial-time algorithmic procedure for generating essentially controllable graphs, exploring scenarios with multiple leaders, and extending the proposed classifications to other, possibly nonlinear, networked control systems are among other areas of future work.

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