On the existence and uniqueness of solutions to the output regulation equations for periodic exosystems

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ABSTRACT

In this paper we prove that, for a general class of control-affine systems, the output regulation equations are uniquely solvable whenever the exosystem is periodic and the linearized zero-dynamics of the plant does not contain periodic solutions of the same period as those of the exosystem. Our main result can therefore be applied to cases when the linearized zero-dynamics are non-hyperbolic. As an application, we consider the important case of when the exosystem is composed of k-uncoupled harmonic oscillators.

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1. Introduction

Consider the controlled dynamical system

$$\dot{x} = f(x, w) + g(x, w)u$$
$$\dot{w} = s(w) \tag{1}$$
$$e = h(x, w)$$

where $x \in \mathbb{R}^n$ is the state variable, $u \in \mathbb{R}^m$ is the control variable, $e \in \mathbb{R}^p$ is the output variable, and $w \in \mathbb{R}^q$ is an external variable. The mappings $f : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^{n \times m}$, $s : \mathbb{R}^q \to \mathbb{R}^q$, and $h : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^p$, defined possibly only locally about the respective origins, are assumed to be smooth. It is assumed that $f(0, 0) = 0$, $s(0) = 0$, and $h(0, 0) = 0$. We make the simplifying assumption that $m = p$, i.e., the input–output system is square. The dynamics of the variable w are referred to as the *exosystem* and represent external disturbances and/or a generator of reference trajectories for the state variable x.

The *output regulation problem* for (1) is to find a feedback control $u = \alpha(x, w)$, with $\alpha(0, 0) = 0$, such that

$$\dot{x} = f(x, 0) + g(x, 0)\alpha(x, 0)$$

has $x = 0$ as an exponentially stable equilibrium, and for each sufficiently small initial condition (x_0, w_0) the solution of (1) with $u = \alpha(x, w)$ satisfies

$$\lim_{t \to \infty} e(t) = 0.$$

It is known [1,2] that, under mild assumptions on the plant and exosystem dynamics, the output regulation problem is solvable if and only if there exist smooth mappings $\pi : \Omega \to \mathbb{R}^n$, with $\pi(0) = 0$, and $\kappa : \Omega \to \mathbb{R}^m$, with $\kappa(0) = 0$, both defined in a neighborhood $\Omega \subseteq \mathbb{R}^q$ of $w = 0$, solving the Francis–Byrnes–Isidori (FBI) equations

$$\frac{\partial \pi}{\partial w}(w)s(w) = f(\pi(w), w) + g(\pi(w), w)\kappa(w) \tag{2}$$
$$0 = h(\pi(w), w).$$

In [1], a subclass of system (1) given by

$$f(x, w) = f_0(x) + p(x)w$$
$$g(x, w) = g_0(x)$$
$$h(x, w) = h_0(x) + q(w)$$

was considered and it was shown that, under a well-defined relative degree assumption, solving the FBI equations can be reduced to solving an invariant manifold PDE for the zero-dynamics of the plant and the exosystem. This reduction principle was generalized in [2] to the general system (1). Specifically, and referring to [2] for the full details, suppose that the composite system (1) has a well-defined vector relative degree $(r_1, \ldots, r_p)$ at $(x, w) = (0, 0)$, let $r = r_1 + \cdots + r_p$, and let $d = n - r$. Then the zero-dynamics of the composite system (1) take the form

$$\dot{z} = \tilde{\zeta}(z, w)$$
$$\dot{w} = s(w) \tag{3}$$

where $z \in \mathbb{R}^d$, and because $m = p$ they are uniquely determined up to a coordinate transformation. As shown in [2], the FBI equations

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are then solvable if there exists an invariant manifold for (3) of the form $\{(z, w) : z = \tilde{\pi}(w)\}$, where $\tilde{\pi}$ is a smooth mapping defined in a neighborhood of $w = 0$. Consequently, one can solve the output regulation problem by finding a local solution $\tilde{\pi}$ to the invariant manifold PDE

$$\frac{\partial \tilde{\pi}}{\partial w}(w)s(w) = \tilde{\zeta}(\tilde{\pi}(w), w). \tag{4}$$

Because $m = p, \kappa$ is unique if it exists, and if $\tilde{\pi}$ is unique then π is also unique. Uniqueness here should be understood up to a coordinate transformation.

With regard to solving (4), it is a standard assumption in the output regulation problem [1,2] that the exosystem has $w = 0$ as a non-attractive Liapunov stable equilibrium. Consequently, the eigenvalues of the matrix $\frac{\partial s}{\partial w}(0)$ will lie on the imaginary axis, and thus if the eigenvalues of the matrix $\frac{\partial \tilde{\zeta}}{\partial z}(0, 0)$ lie off the imaginary axis, i.e., the hyperbolic case, then by the well-known center manifold theorem [3] one can deduce that a solution (not necessarily unique) to the PDE (4) exists. In the case of real-analytic data and two dimensional exosystems, applying the main result in [4], it was deduced in [5] that (4) in fact has a unique solution, is analytic and the invariant manifold is generated by a one-parameter family of periodic solutions.

The purpose of this paper is to extend the results in [5] by considering exosystems generating periodic trajectories and prove the existence and uniqueness of $\tilde{\pi}$ in the possibly non-hyperbolic case. A key ingredient in the proof of our main result (Theorem 2.2) is the triangular structure of (3) which simplifies the study of the flow of (3). This extra structure allows us to avoid the use of the center manifold theorem and allows the possibility of non-hyperbolic zero-dynamics. In fact, as shown in the example in [6], hyperbolicity of the zero-dynamics is not necessary for the existence of $\tilde{\pi}$. As was the case in the example in [6], the key property to deduce the existence of $\tilde{\pi}$ is that the exosystem does not generate trajectories of the same period as those of the linear dynamical system $\dot{z} = \frac{\partial \tilde{\zeta}}{\partial z}(0, 0)z$. This property is analogous to the condition needed for the solvability of the linear output regulation problem proved by Hautus [7], namely, that the eigenvalues of the exosystem do not intersect the eigenvalues of the zero-dynamics, i.e., do not intersect the transmission zeros of the plant. Hence, our result can be seen as a nonlinear version of Hautus' test.

As an application of our main result on output tracking, we consider the case when the exosystem is composed of k-uncoupled harmonic oscillators. This class of exosystem is used widely in applications as it can be used to model sinusoidal disturbances and simultaneously generate sinusoidal output reference trajectories.

This paper is organized as follows. In Section 2 we state and prove our main results. In Section 3 we apply our main result to the problem of tracking periodic trajectories generated by k-uncoupled harmonic oscillators. In Section 4 we present two examples illustrating our main results. We end the paper with some concluding remarks and avenues of future research.

2. Main results

Rewrite the dynamical system (3) as

$$\begin{aligned} \dot{z} &= Bz + \zeta(z, w) \\ \dot{w} &= s(w) \end{aligned} \tag{5}$$

where now $\frac{\partial \zeta}{\partial z}(0, 0) = 0, \zeta(0, 0) = 0, z \in \mathbb{R}^d$, and $w \in \mathbb{R}^q$. The flow of (5) will be denoted by $(t, z, w) \mapsto (\phi(t, z, w), \psi(t, z, w))$. We have that $(\phi(t, 0, 0), \psi(t, 0, 0)) = (0, 0)$ for all t because $(0, 0)$ is an equilibrium solution. We note that since the w-equation in (5) is independent of z, $\psi(t, z, w)$ is

actually independent of z, and thus we drop the dependence of ψ on z. For convenience write the linear dynamical system

$$\dot{z} = Bz \tag{6}$$

for future reference. Also, we note that the invariant manifold PDE (4) now takes the form

$$\frac{\partial \tilde{\pi}}{\partial w}(w)s(w) = B\tilde{\pi}(w) + \zeta(\tilde{\pi}(w), w). \tag{7}$$

To state our main results, we need the following standard definition.

Definition 2.1. Let $\dot{x} = f(x)$ denote a dynamical system where $f : E \rightarrow \mathbb{R}^n$ is a vector field defined on an open set $E \subset \mathbb{R}^n$ and let ϕ_t denote the flow of f. A set $S \subset E$ is said to be invariant with respect to the flow ϕ_t if $\phi_t(S) \subset S$ for all $t \in \mathbb{R}$.

Throughout the paper we assume that $T > 0$ is a time parameter. We now state our main results.

Theorem 2.1. Consider system (1) and assume that the pair $(\frac{\partial f}{\partial x}(0, 0), g(0, 0))$ is stabilizable. Suppose that there exists a neighborhood $\mathcal{W}$ of $w = 0$ such that the solutions of the exosystem initiating in $\mathcal{W}$ are T-periodic, and any neighborhood $\mathcal{W}' \subset \mathcal{W}$ of $w = 0$ contains an open invariant subset containing $w = 0$. Suppose that (1) has a well-defined relative degree at $(x, w) = (0, 0)$ and let (5) denote the zero-dynamics of (1). If the linear system (6) has no T-periodic solution other than the zero solution then the FBI equations (2) have a smooth and unique solution.

We note that in Theorem 2.1, the invariance assumption implies that the exosystem has $w = 0$ as a non-attractive Liapunov stable equilibrium. Theorem 2.1 follows from our discussion in Section 1 and the following theorem.

Theorem 2.2. Consider a dynamical system of the form (5) having an equilibrium at the origin $(z, w) = (0, 0)$. Suppose that there exists a neighborhood $\mathcal{W}$ of $w = 0$ such that solutions to $\dot{w} = s(w)$ initiating in $\mathcal{W}$ are T-periodic, and any neighborhood $\mathcal{W}' \subset \mathcal{W}$ of $w = 0$ contains an open invariant subset containing $w = 0$. If the linear system (6) has no T-periodic solution other than the zero solution then (7) has a smooth and unique solution defined locally about $w = 0$ and whose graph defines an invariant manifold for (3).

Proof. For $w \in \mathcal{W}$, consider the equation

$$z = \phi(T, z, w) \tag{8}$$

and note that from the variations of constant formula

$$\phi(T, z, w) = e^{BT}z + \int_0^T e^{B(T-s)}\zeta(\phi(s, z, w), \psi(s, w)) \, ds.$$

If (z, w) is a solution to (8) then clearly $(\phi(t, z, w), \psi(t, w))$ is T-periodic. Hence, consider the mapping

$$F(z, w) = -z + \phi(T, z, w)$$

or equivalently,

$$F(z, w) = (e^{BT} - I)z + \int_0^T e^{B(T-s)}\zeta(\phi(s, z, w), \psi(s, w)) \, ds.$$

We note that because $(0, 0)$ is an equilibrium solution of (5), by the well-known dependence of initial conditions of a smooth dynamical system [3], the mapping F is well-defined in some neighborhood of $(0, 0)$. It is clear that

$$F(0, 0) = 0.$$

Now

$$\frac{\partial F}{\partial z}(z, w) = (e^{BT} - I) + \int_0^T \left[e^{B(T-s)} \frac{\partial \zeta}{\partial z}(\phi(s, z, w), \psi(s, w)) \right.$$
$$\left. \times \frac{\partial \phi}{\partial z}(s, z, w) \right] ds$$

and because $\phi(t, 0, 0) = 0$, $\psi(t, 0) = 0$, for all t, and $\frac{\partial \zeta}{\partial z}(0, 0) = 0$, we have that

$$\frac{\partial F}{\partial z}(0, 0) = (e^{BT} - I).$$

The assumption that (6) does not have any non-zero T-periodic solutions implies that $(e^{BT} - I)$ is invertible. Indeed, if $z_0 \neq 0$ satisfies $(e^{BT} - I)z_0 = 0$ then $z(t) = e^{Bt}z_0$ is a non-zero T-periodic solution of (6), which is a contradiction. Therefore, by the implicit function theorem, there is a unique mapping $\tilde{\pi} : \mathcal{W}' \to \mathcal{U}'$, where $\mathcal{W}' \times \mathcal{U}'$ is a neighborhood of $(0, 0)$, such that $\tilde{\pi}(0) = 0$, and

$$F(\tilde{\pi}(w), w) = 0$$

and $\tilde{\pi}$ is smooth. It follows that $t \mapsto (\phi(t, \tilde{\pi}(w), w), \psi(t, w))$ is a T-periodic solution of (5) for all $w \in \mathcal{W}'$, and $z = \tilde{\pi}(w)$ is the unique initial condition for the z-component of (5) which results in a T-periodic solution for a given initial condition w of the w-component of (5). By continuity, we can assume without loss of generality that

$$\frac{\partial F}{\partial z}(\tilde{\pi}(w), w)$$

is also invertible for all $w \in \mathcal{W}'$.

Now, from the relation

$$0 = F(\tilde{\pi}(w), w) = -\tilde{\pi}(w) + \phi(T, \tilde{\pi}(w), w)$$

and the chain rule we obtain that

$$\left[-I + \frac{\partial \phi}{\partial z}(T, \tilde{\pi}(w), w) \right] \frac{\partial \tilde{\pi}}{\partial w} + \frac{\partial \phi}{\partial w}(T, \tilde{\pi}(w), w) = 0. \quad (9)$$

Multiplying both sides of (9) by $s(w)$ and rearranging we obtain

$$\frac{\partial \tilde{\pi}}{\partial w}(w)s(w) = \frac{\partial \phi}{\partial z}(T, \tilde{\pi}(w), w)\frac{\partial \tilde{\pi}}{\partial w}(w)s(w)$$
$$+ \frac{\partial \phi}{\partial w}(T, \tilde{\pi}(w), w)s(w). \quad (10)$$

Consider now the linearization of (5) along any solution $(\phi(t, z, w), \psi(t, w))$, which is given by

$$\frac{d}{dt}\xi(t, z, w) = A(t, z, w)\xi(t, z, w) \quad (11)$$

where $A(t, z, w)$ is the Jacobian of the right-hand-side of (5) evaluated along the solution $(\phi(t, z, w), \psi(t, w))$. Clearly, $\xi(t, z, w) = (\frac{d}{dt}\phi(t, z, w), \frac{d}{dt}\psi(t, w))$ is a solution of (11), and therefore if the initial condition of (11) at $t = 0$ is

$$\xi(0, z, w) = \xi(0, \tilde{\pi}(w), w) := \begin{bmatrix} B\tilde{\pi}(w) + \zeta(\tilde{\pi}(w), w) \\ s(w) \end{bmatrix}$$

then the corresponding solution $\xi(t, \tilde{\pi}(w), w)$ is T-periodic. If $\Phi(t, z, w)$ denotes the fundamental matrix of the linear system (11), then T-periodicity of $\xi(t, \tilde{\pi}(w), w)$ implies that

$$\xi(0, \tilde{\pi}(w), w) = \Phi(T, \tilde{\pi}(w), w)\xi(0, \tilde{\pi}(w), w). \quad (12)$$

It is known that $\Phi(t, z, w)$ is the derivative of the mapping $(z, w) \mapsto (\phi(t, z, w), \psi(t, z))$ [3, p. 83], that is

$$\Phi(t, z, w) = \begin{bmatrix} \frac{\partial \phi}{\partial z}(t, z, w) & \frac{\partial \phi}{\partial w}(t, z, w) \\ 0 & \frac{\partial \psi}{\partial w}(t, w) \end{bmatrix}.$$

Therefore, from (10) and the fact that $\psi(t, w)$ is T-periodic, we can write that also

$$\begin{bmatrix} \frac{\partial \tilde{\pi}}{\partial w}(w)s(w) \\ s(w) \end{bmatrix} = \Phi(T, \tilde{\pi}(w), w) \begin{bmatrix} \frac{\partial \tilde{\pi}}{\partial w}(w)s(w) \\ s(w) \end{bmatrix}. \quad (13)$$

Combining (12)–(13) it follows that

$$0 = \left[\frac{\partial \phi}{\partial z}(T, \tilde{\pi}(w), w) - I \right]$$
$$\times \left[\frac{\partial \tilde{\pi}}{\partial w}(w)s(w) - (B\tilde{\pi}(w) + \zeta(\tilde{\pi}(w), w)) \right].$$

But

$$\frac{\partial F}{\partial z}(z, w) = -I + \frac{\partial \phi}{\partial z}(T, z, w)$$

and therefore

$$\frac{\partial F}{\partial z}(\tilde{\pi}(w), w)\left[\frac{\partial \tilde{\pi}}{\partial w}(w)s(w) - (B\tilde{\pi}(w) + \zeta(\tilde{\pi}(w), w)) \right] = 0.$$

Now because $\frac{\partial F}{\partial z}(\tilde{\pi}(w), w)$ is invertible for all $w \in \mathcal{W}'$ it follows that

$$\frac{\partial \tilde{\pi}}{\partial w}(w)s(w) = B\tilde{\pi}(w) + \zeta(\tilde{\pi}(w), w)$$

for all $w \in \mathcal{W}'$. By shrinking $\mathcal{W}'$ if necessary, we can assume that $\mathcal{W}'$ is an invariant set for $\dot{w} = s(w)$. Hence,

$$\{(z, w) : z = \tilde{\pi}(w), \ w \in \mathcal{W}'\}$$

is an invariant manifold for (5).

Proving that $\tilde{\pi}$ is the unique solution of (7) on $\mathcal{W}'$ is straightforward. Suppose that $\bar{\pi}$ solves (7) on $\mathcal{W}'$. Let $w(t)$ be a solution of $\dot{w} = s(w)$ initiating in $\mathcal{W}'$, and thus of period T. Then clearly, the curve $\bar{z}(t) = \bar{\pi}(w(t))$ is also T-periodic. Using the fact that $\bar{\pi}$ is a solution of (7) on $\mathcal{W}'$, a direct computation using the chain rule shows that

$$\frac{d}{dt}\bar{z}(t) = B\bar{z}(t) + \zeta(\bar{z}(t), w(t)).$$

In other words, the curve $t \mapsto (\bar{z}(t), w(t))$ is a T-periodic solution of (5). Recalling that $\tilde{\pi}(w(0))$ is the unique initial condition for the z-component of (5) which results in a T-periodic solution for the initial condition $w(0)$, it follows that $\bar{z}(0) = \tilde{\pi}(w(0))$, i.e., $\bar{\pi}(w(0)) = \tilde{\pi}(w(0))$. This holds for all initial conditions $w(0) \in \mathcal{W}'$, and thus proves uniqueness of $\tilde{\pi}$ on $\mathcal{W}'$. This completes the proof. $\square$

The interesting case when Theorem 2.2 is applicable is when B has eigenvalues on the imaginary axis, otherwise the *existence* part of the theorem is a direct consequence of the center manifold theorem. However, even when the center manifold theorem is applicable, Theorem 2.2 gives *uniqueness* of solutions. In general, the manifold $z = \tilde{\pi}(w)$ in Theorem 2.2 is a submanifold of every center manifold of (5) since every center manifold contains periodic trajectories sufficiently close to the origin [8].

3. Application: k-uncoupled harmonic oscillators

The assumption in Theorem 2.1 that the exosystem generates periodic trajectories of a single period T is a significant restriction. On the other hand, there is an important class of exosystems that satisfies this condition, namely, an exosystem consisting of k-uncoupled harmonic oscillators with rational frequencies $\omega_1, \ldots, \omega_k \in \mathbb{Q}$. This type of exosystem can be used to model sinusoidal disturbances of say frequencies $\omega_1, \ldots, \omega_r$, and the

remaining frequencies $\omega_{r+1}, \ldots, \omega_k$ can be used to generate periodic reference trajectories for the state variable.

Consider then the exosystem

$$\dot{w} = Sw \tag{14}$$

where $S = \text{diag}(S_1, S_2, \ldots, S_k)$ and

$$S_i = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix}$$

for $i = 1, \ldots, k$. Suppose that the frequencies $\omega_1, \ldots, \omega_k$ are rational, say $\omega_i = \frac{a_i}{b_i}$, with $\gcd(a_i, b_i) = 1$, and decompose $a_i = 2^{\ell_i} c_i$ for unique $\ell_i \in \{0, 1, \ldots\}$ and positive integers c_i. Let $b = \text{lcm}(b_1, \ldots, b_k)$, let $c = \gcd(c_1, \ldots, c_k)$, and let $\ell = \min(\ell_1, \ldots, \ell_k)$. Then

$$T^* = \frac{b}{2^{\ell-1} c} \pi$$

is the minimum real number such that for all $i = 1, \ldots, k$

$$\omega_i T^* = 2\pi n_i$$

for some positive integers n_i. In other words, T^* is the period of the k-uncoupled oscillators, and therefore the period of (14). Of course, it is possible to choose the initial condition of (14) such that the resulting trajectory is periodic of period $T' < T^*$, say by setting the initial condition of a subset of the k oscillators to the origin. However, elementary considerations show that necessarily $T^* = NT'$ for some positive integer N. With this in mind we have the following corollary to Theorem 2.1.

Theorem 3.1. *Consider the control system* (1) *and suppose that it has a well-defined relative degree at* $(x, w) = (0, 0)$. *Assume that the pair* $(\frac{\partial f}{\partial x}(0, 0), g(0, 0))$ *is stabilizable. Suppose that the exosystem is given by* (14) *with* $\omega_1, \ldots, \omega_k \in \mathbb{Q}$. *Suppose that B does not have 0 as an eigenvalue and let* $\{\pm\nu_1 i, \ldots, \pm\nu_r i\}$ *denote the eigenvalues of B on the imaginary axis. If*

$$\left\{ \frac{2\pi}{\nu_1}, \ldots, \frac{2\pi}{\nu_r} \right\} \cap \left\{ \frac{1}{N} T^* : N = 1, 2, \ldots \right\} = \emptyset \tag{15}$$

then the associated FBI equations have a smooth and unique solution. Consequently, the output regulation problem is solvable.

Proof. As was discussed above, every solution of (14) is T^* periodic. For any open set $\mathcal{W}$ containing $w = 0$, it is not hard to see that there exists an open $\mathcal{W}' \subset \mathcal{W}$ that is invariant under the flow of (14) and contains $w = 0$. For example, $\mathcal{W}'$ can be taken as the Cartesian product of k open discs in $\mathbb{R}^2$ with the ith disc being an invariant set for the ith harmonic oscillator in (14).

The only possible periodic trajectories for (6) are those that initiate in the invariant subspaces of B associated with the eigenvalues on the imaginary axis [3], and by assumption these subspaces all correspond to non-zero eigenvalues. Now, any non-zero periodic trajectory of (6) will have a period that is a positive integer multiple of $\frac{2\pi}{\nu_j}$ for some $j \in \{1, \ldots, r\}$, say $N\frac{2\pi}{\nu_j}$. By assumption, $N\frac{2\pi}{\nu_j} \neq T^*$ and therefore (6) does not contain a non-zero T^*-periodic trajectory. The claim now follows by Theorem 2.1. □

Remark 3.1. It is straightforward to verify that (15) is satisfied if and only if the numbers $\frac{b\nu_1}{2^\ell c}, \ldots, \frac{b\nu_r}{2^\ell c}$ are not (positive) integers. It is interesting to compare (15) with the resonance condition given in [9, Lemma 5.2] for the solvability of the output regulation problem. Specifically, in [9, Lemma 5.2], the condition

$$\text{rank} \begin{bmatrix} \frac{\partial f}{\partial x}(0, 0) - \lambda I & g(0, 0) \\ \frac{\partial h}{\partial x}(0, 0) & 0_{p \times m} \end{bmatrix} = n + p \tag{16}$$

for all $\lambda = \lambda_{i_1} + \cdots + \lambda_{i_\beta}$ and all $\beta = 1, 2 \ldots$, where λ_{i_j} are eigenvalues of $\frac{\partial s}{\partial w}(0)$, is given for the formal solvability of the invariant manifold PDE associated to the output regulation problem. Condition (16) is difficult to verify in practice as it requires the verification of (16) on an *infinite* number of λ's.

4. Examples

In this section we present two examples illustrating the applicability of Theorem 3.1 when the center manifold theorem or the results in [5] are not applicable. The first example, taken from [10,6], was a motivation for the current paper.

Example 4.1. Consider the classical inverted pendulum system with the addition of a second cart connected to the first cart by a spring having spring constant K. As in the classical system, a freely hanging pendulum is attached to the first cart. The first cart is actuated by a horizontal force u and both carts have only horizontal motion in-line with the applied force. Both carts have equal mass M, the length of the pendulum rod is ℓ and has mass m. The equations of motion of the system are

$$(M + m)\ddot{x}_1 + m\ell(\ddot{\theta} \cos\theta - \dot{\theta}^2 \sin\theta) = u + K(x_2 - x_1)$$
$$m\ell\ddot{x}_1 \cos\theta + m\ell^2\ddot{\theta} = mg\ell \sin\theta$$
$$M\ddot{x}_2 = -K(x_2 - x_1)$$

where x_1 is the position of the first cart, x_2 is the position of the second cart, θ is the angle the pendulum makes with the vertical-up position, and g is the acceleration due to gravity. As output we are interested in the position x_1 of the actuated cart and consider the output regulation problem with a sinusoidal reference signal. We therefore let $e = h(x, w) = x_1 - w_1$ and choose a two-dimensional exosystem consisting of a harmonic oscillator with frequency ω, i.e., $k = 1$ in (14). It can be verified that the zero-dynamics of the system can be represented in the form [6]

$$\dot{z}_1 = z_2 \qquad\qquad \dot{w}_1 = \omega w_2$$
$$\dot{z}_2 = \frac{\omega^2}{\ell} w_1 \cos(z_1) + \frac{g}{\ell} \sin(z_1) \quad \dot{w}_2 = -\omega w_1$$
$$\dot{z}_3 = z_4$$
$$\dot{z}_4 = \frac{K}{M}(w_1 - z_3).$$

The eigenvalues of the linear part of the z-dynamics are $\pm\sqrt{g/\ell}$ and $\pm\sqrt{K/M}i$. Therefore one cannot deduce the solvability of the associated invariant manifold PDE from the center manifold theorem or from [5]. In this case we have that $T^* = 2\pi/\omega$ and $\nu_1 = \sqrt{K/M}$. Therefore, if $\sqrt{K/M} \neq N\omega$ for all $N = 1, 2, \ldots$, by Theorem 3.1 the associated FBI equations are solvable, as proved in [6]. Moreover, by Theorem 3.1, the FBI equations have a unique (local) solution.

Example 4.2. Consider the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = a(x) + b(x)u$$
$$\dot{x}_3 = x_1 + \frac{1}{9}x_4 + x_2 x_4$$
$$\dot{x}_4 = -4x_3 + x_4^3 + x_1 x_2 + x_1 x_3 \tag{17}$$
$$y = x_1 - w_1$$
$$\dot{w}_1 = w_2$$
$$\dot{w}_2 = -w_1$$

where $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ is the state, $u \in \mathbb{R}$ is the control, and a and b are smooth functions with $a(0) = 0$ and $b(0) \neq 0$.

A system similar to (17) was considered in [11]. System (17) has a well-defined relative degree at the origin and its zero-dynamics can be written as

$$\dot{z}_1 = \frac{1}{9}z_2 + w_1 + w_2 z_2$$

$$\dot{z}_2 = -4z_1 + z_2^3 + w_1 w_2 + w_1 z_1$$

$$\dot{w}_1 = 2w_2$$

$$\dot{w}_2 = -2w_1.$$

The eigenvalues of the linear system $\dot{z}_1 = \frac{1}{9}z_2, \dot{z}_2 = -4z_1$ are $\pm\frac{2}{3}i$, and the eigenvalues of the exosystem are $\pm 2i$. Therefore one cannot deduce the solvability of the associated invariant manifold PDE from the center manifold theorem or from [5]. In this case we have that $T^* = \pi$ and $2\pi\frac{3}{2} \neq \frac{\pi}{N}$ for all $N = 1, 2, \ldots$. Therefore, by Theorem 3.1 the associated FBI equations are uniquely solvable.

5. Conclusion

In this paper we proved that, for a general class of nonlinear control-affine systems with a well-defined relative degree at the origin, the output regulation equations are uniquely solvable whenever the exosystem does not generate periodic trajectories of the same period as those of the linearized zero-dynamics. In view of the current results, in a future paper we intend to extend the numerical algorithm presented in [5] for exosystems of dimension greater than two.

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