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Laplacian controllability classes for threshold graphs

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A B S T R A C T

Let $\mathcal{G}$ be a graph on n vertices with Laplacian matrix $\mathbf{L}$ and let $\mathbf{b}$ be a binary vector of length n. The pair $(\mathbf{L}, \mathbf{b})$ is *controllable* if the smallest $\mathbf{L}$-invariant subspace containing $\mathbf{b}$ is of dimension n. The graph $\mathcal{G}$ is called *essentially controllable* if $(\mathbf{L}, \mathbf{b})$ is controllable for every $\mathbf{b} \notin \ker(\mathbf{L})$, *completely uncontrollable* if $(\mathbf{L}, \mathbf{b})$ is uncontrollable for every $\mathbf{b}$, and *conditionally controllable* if it is neither essentially controllable nor completely uncontrollable. In this paper, we completely characterize the graph controllability classes for threshold graphs. We first observe that the class of threshold graphs contains no essentially controllable graph. We prove that a threshold graph is completely uncontrollable if and only if its Laplacian matrix has a repeated eigenvalue. In the process, we fully characterize the set of conditionally controllable threshold graphs.

1. Introduction

Consider the single-input linear control system

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$$\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{b}u(t) \tag{1}$$

where $\mathbf{F} \in \mathbb{R}^{n\times n}$, $\mathbf{b} \in \mathbb{R}^n$, $\mathbf{x}(t) \in \mathbb{R}^n$, and $u(t) \in \mathbb{R}$. If for each $\mathbf{x}_0 \in \mathbb{R}^n$ there exists a *control signal* $u : \mathbb{R} \to \mathbb{R}$ such that the trajectory of (1) with initial condition $\mathbf{x}(0) = \mathbf{x}_0$ reaches the origin in finite time, then the pair $(\mathbf{F}, \mathbf{b})$ is called *controllable*. It is well-known that $(\mathbf{F}, \mathbf{b})$ is controllable if and only if the smallest $\mathbf{F}$-invariant subspace containing $\mathbf{b}$, denoted by $\langle \mathbf{F}; \mathbf{b}\rangle$, has full dimension n [12]. Although controllability of linear systems is a well developed subject, the problem has drawn recent interest due to applications in networked dynamical systems and distributed control. Specifically, the case where $\mathbf{F}$ is the Laplacian matrix $\mathbf{L}$ of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and $\mathbf{b} \in \{0, 1\}^n$ is a binary vector, has drawn a great deal of attention in recent years [20,18,11,10,17,15]. In engineering applications, the vertices $\mathcal{V}_\mathbf{b} := \{v_i \in \mathcal{V} \mid (\mathbf{b})_i = 1\}$ are seen as *leader* agents and influence the remaining *follower* agents $\mathcal{V}\backslash\mathcal{V}_\mathbf{b}$ through the control signal $u : \mathbb{R} \to \mathbb{R}$ and the connectivity of the communication network defined by the graph $\mathcal{G}$. A major problem of interest is to characterize the controllability properties of $(\mathbf{L}, \mathbf{b})$ in terms of the topological properties of $\mathcal{G}$ as $\mathbf{b}$ is allowed to vary within the set $\{0, 1\}^n$ of binary vectors. The reason for studying (1) with the Laplacian matrix is that it serves as a benchmark for studying consensus algorithms [16], and moreover, the problem is of independent interest since its characterization reveals valuable information about the eigenvectors of the Laplacian and adjacency matrices of graphs [5,4,3].

In this paper, we study the topological obstructions to controllability via the notion of *graph controllability classes*, recently introduced in [1].

Definition 1.1. Let $\mathcal{G}$ be a connected graph with Laplacian matrix $\mathbf{L}$. Then $\mathcal{G}$ is called

(i) *essentially controllable* on $\{0, 1\}^n$ if $(\mathbf{L}, \mathbf{b})$ is controllable for every $\mathbf{b} \in \{0, 1\}^n\backslash \ker(\mathbf{L})$;
(ii) *completely uncontrollable* on $\{0, 1\}^n$ if $(\mathbf{L}, \mathbf{b})$ is uncontrollable for every $\mathbf{b} \in \{0, 1\}^n$; and
(iii) *conditionally controllable* on $\{0, 1\}^n$ if it is neither essentially controllable nor completely uncontrollable on $\{0, 1\}^n$.

For each integer $n \geq 2$, let a_n be the number of asymmetric connected graphs and let e_n be the number of essentially controllable graphs, of order n. It is known that an essentially controllable graph of order larger than two must be asymmetric. On the other hand, the block graphs of Steiner triple systems generate asymmetric graphs of arbitrarily large order that are completely uncontrollable. However, it is conjectured that $\lim_{n\to\infty} e_n/a_n = 1$.

In this paper, we consider *threshold* graphs and show that the presence of a repeated eigenvalue is a necessary condition for complete uncontrollability and in the process completely classify the set of conditionally controllable threshold graphs. Threshold graphs were introduced in [2] and in [9], and their interesting properties has led to a large body of literature, see [6] and [13], and references therein. In applications, threshold graphs appear as models of social networks [19], in the problem of synchronizing parallel computer processes, cyclic scheduling problems, and problems in psychology [13]. The results of this paper rely on the spectral properties of the Laplacian for threshold graphs characterized in [8,14] in terms of the degree sequence and which enables us to explicitly determine the only threshold graphs with simple eigenvalues.

1.1. Notation

In this section we establish some notation used throughout the paper. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph (undirected, unweighted, no loops or multiple edges) with vertex set $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$. The set of vertices adjacent to $v \in \mathcal{V}$ will be denoted by $\mathcal{N}(v) := \{w \in \mathcal{V} \mid \{v, w\} \in \mathcal{E}\}$ and the degree of v will be denoted by $d_v := |\mathcal{N}(v)|$. Accepting a slight abuse of notation, we denote an edge by (v, w) with the understanding that this pair is unordered. The adjacency matrix of $\mathcal{G}$ will be denoted by $\mathbf{A}$, the diagonal degree matrix by $\mathbf{D}$, and the Laplacian matrix of $\mathcal{G}$ by $\mathbf{L} = \mathbf{D} - \mathbf{A}$. The Laplacian $\mathbf{L}$ is symmetric and positive semi-definite, and thus its eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are real and non-negative. The all ones vector $\mathbf{1}_n := [1\ 1\ \cdots\ 1]^T$ is an eigenvector of $\mathbf{L}$ with eigenvalue $\lambda_1 = 0$, and if $\mathcal{G}$ is connected then $\lambda_1 = 0$ is a simple eigenvalue. We assume throughout that $\mathcal{G}$ is connected so that $0 < \lambda_2$. For our purposes, by the eigenvalues (eigenvectors) of a graph $\mathcal{G}$ we mean the eigenvalues (eigenvectors) of its Laplacian matrix $\mathbf{L}$.

Finally, we denote by $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ the standard basis of $\mathbb{R}^n$, and if $\mathbf{v} \in \mathbb{R}^n$ we denote by $(\mathbf{v})_a$ the a-th component of $\mathbf{v}$.

2. Threshold graphs

We recall the definition of threshold graphs from [8]. We start with a single vertex v_1 and let $\mathcal{G}_1 = (\{v_1\}, \emptyset)$. Suppose now that $\mathcal{G}_k = (\mathcal{V}_k, \mathcal{E}_k)$ has been defined for some $k \geq 1$. Then $\mathcal{G}_{k+1}$ is obtained from $\mathcal{G}_k$ by adding a new vertex v_{k+1} and defining the new edge set as either $\mathcal{E}_{k+1} = \mathcal{E}_k \oplus \{v_{k+1}\} := \mathcal{E}_k$ or $\mathcal{E}_{k+1} = \mathcal{E}_k \otimes \{v_{k+1}\} := \mathcal{E}_k \cup \{(v_1, v_{k+1}), \dots, (v_k, v_{k+1})\}$. In other words, the $\oplus$ operation simply adds the vertex v_{k+1} to the graph $\mathcal{G}_k$ without connecting it to any vertex of $\mathcal{G}_k$ and the $\otimes$ operation connects v_{k+1} to each vertex of $\mathcal{G}_k$. Henceforth, we assume that the vertex set $\mathcal{V} = \{v_1, \dots, v_n\}$ of a threshold graph is labeled according to the above inductive construction. Following [7], we associate with a threshold graph $\mathcal{G}$ of order n a binary *creation* sequence $T_{\mathcal{G}} \in \{0, 1\}^n$ defined as $T_{\mathcal{G}}(i) = 0$ (respectively $T_{\mathcal{G}}(i) = 1$) if the $\oplus$ operation (respectively $\otimes$) was used when adding vertex v_i, for all $i \in \{1, \dots, n\}$. We note that a threshold graph is connected if and only if $T_{\mathcal{G}}(n) = 1$.

Let $\mathcal{G}$ be a threshold graph of order n and let $D(\mathcal{G}) = (d_1, \dots, d_n)$ be the degree sequence of $\mathcal{G}$, i.e., $|\mathcal{N}(v_i)| = d_i$. Let $\delta_1 < \delta_2 < \cdots < \delta_s$, where $s \leq n$, be the distinct degrees appearing in $D(\mathcal{G})$ and let n_i be the number of times δ_i appears in $D(\mathcal{G})$. With a slight abuse of notation we write $D(\mathcal{G}) = (\delta_1^{n_1}, \dots, \delta_s^{n_s})$. Similarly, by $\Lambda(\mathcal{G}) = (\mu_1^{m_1}, \dots, \mu_h^{m_h})$ we denote the spectrum sequence of $\mathbf{L}$ where $\mu_1 < \mu_2 < \cdots < \mu_h$ are the distinct *non-zero* eigenvalues of $\mathbf{L}$, where m_i is the algebraic multiplicity of μ_i. With this notation we have the following theorem [8].

Theorem 2.1. *Let $\mathcal{G}$ be a connected threshold graph with degree and spectrum sequences $D(\mathcal{G}) = (\delta_1^{n_1}, \ldots, \delta_s^{n_s})$ and $\Lambda(\mathcal{G}) = (\mu_1^{m_1}, \ldots, \mu_h^{m_h})$, respectively. Then $s = h$ and:*

1. *If h is odd, say $h = 2r + 1$, then*

$$\mu_i = \begin{cases} \delta_i, & i = 1, \ldots, r, \\ \delta_i + 1, & i = r+1, \ldots, h \end{cases} \quad \text{and} \quad m_i = \begin{cases} n_i - 1, & i = r+1, \\ n_i, & \text{otherwise}. \end{cases}$$

2. *If h is even, say $h = 2r$, then*

$$\mu_i = \begin{cases} \delta_i, & i = 1, \ldots, r, \\ \delta_i + 1, & i = r+1, \ldots, h \end{cases} \quad \text{and} \quad m_i = \begin{cases} n_i - 1, & i = r, \\ n_i, & \text{otherwise}. \end{cases}$$

3. Controllability classes for threshold graphs

We start by proving the following straightforward consequence of the definition of threshold graphs.

Lemma 3.1. *Any threshold graph has at least one non-trivial graph automorphism.*

Proof. The result is trivial if the order of the graph is $n = 2$ so assume that $n > 2$. Consider the vertices v_1 and v_2. By the inductive construction of threshold graphs, any vertex $v \in \mathcal{V} \backslash \{v_1, v_2\}$ is adjacent to v_1 if and only if it is adjacent to v_2. Hence, the element of the automorphism group that interchanges v_1 and v_2 and fixes the rest of the graph vertices is a nontrivial graph automorphism. □

We next prove that there exists no essentially controllable graphs but first we need the following [1].

Theorem 3.1. *Let $\mathcal{G}$ be a connected graph with Laplacian matrix $\mathbf{L}$ and let $\mathbf{b}$ be a binary vector not equal to the zero or all ones vector. Suppose that there are positive integers α, β such that for each $v_i \in \mathcal{V}_{\mathbf{b}}$, we have $\alpha = |\mathcal{N}(v_i) \cap \mathcal{V} \backslash \mathcal{V}_{\mathbf{b}}|$, and for each $v_j \in \mathcal{V} \backslash \mathcal{V}_{\mathbf{b}}$ we have $\beta = |\mathcal{N}(v_j) \cap \mathcal{V}_{\mathbf{b}}|$. Then*

$$\mathbf{L}^2 \mathbf{b} = (\alpha + \beta) \mathbf{L} \mathbf{b},$$

and in this case, $\dim\langle \mathbf{L}; \mathbf{b} \rangle = 2$. In particular, $\alpha + \beta$ is an integer eigenvalue of $\mathbf{L}$ with corresponding eigenvector $\mathbf{L}\mathbf{b}$.

Proposition 3.1. *For any connected threshold graph of order $n \geq 2$ we have*

$$\dim\langle \mathbf{L}; \mathbf{e}_n \rangle = 2.$$

In particular, there exists no essentially controllable threshold graph of order $n > 2$.

Proof. Using the notation in Theorem 3.1, if $\mathbf{b} = \mathbf{e}_n$, then clearly $\alpha = n - 1$ and $\beta = 1$, and therefore $\langle \mathbf{L}, \mathbf{e}_n \rangle = \text{span}\{\mathbf{e}_n, \mathbf{L}\mathbf{e}_n\}$. Hence, if $n > 2$ then $(\mathbf{L}, \mathbf{e}_n)$ is uncontrollable. □

We now state one of our main results.

Theorem 3.2. *Let $\mathcal{G}$ be a connected threshold graph of order $n > 2$. Then $\mathcal{G}$ is completely uncontrollable if and only if $\mathbf{L}$ has a repeated eigenvalue.*

Before proving Theorem 3.2, we make the following remark.

Remark 3.1. It is well-known that for a diagonalizable matrix $\mathbf{M} \in \mathbb{R}^{n \times n}$ with distinct eigenvalues $\mu_1, \ldots, \mu_h$, its minimal polynomial is $m(x) = (x - \mu_1)(x - \mu_2) \cdots (x - \mu_h)$. Consequently, $\dim\langle \mathbf{M}; \mathbf{b} \rangle \leq h$ for every $\mathbf{b} \in \mathbb{R}^n$, and thus one direction of Theorem 3.2 is immediate. However, it is not difficult to construct a symmetric matrix $\mathbf{M}$ having distinct eigenvalues such that $\dim\langle \mathbf{M}; \mathbf{b} \rangle < n$ for every $\mathbf{b} \in \{0, 1\}^n$. For example,

$$\mathbf{M} = \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 5 & -3 \\ -1 & -1 & -3 & 5 \end{bmatrix}$$

has distinct eigenvalues $\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_3 = 4$, $\lambda_4 = 8$, and it is readily verified that $(\mathbf{M}, \mathbf{b})$ is uncontrollable for every $\mathbf{b} \in \{0, 1\}^4$.

The proof of Theorem 3.2 relies on the following sequence of results.

Proposition 3.2. *Let $\mathcal{G}$ be a connected threshold graph with creation sequence $T_\mathcal{G}$. Then $\mathcal{G}$ has simple eigenvalues if and only if one of the following holds:*

(i) $T_\mathcal{G} = (0, 1, 0, 1, 0, 1, \ldots, 0, 1)$, *or*
(ii) $T_\mathcal{G} = (0, 0, 1, 0, 1, 0, \ldots, 0, 1)$.

Proof. We claim that if $\mathcal{G}$ satisfies one of the following conditions

a) $T_\mathcal{G}(i) = T_\mathcal{G}(i + 1) = 0$, or
b) $T_\mathcal{G}(i) = T_\mathcal{G}(i + 1) = 1$,

for some $i \geq 2$, then $\mathcal{G}$ has a repeated eigenvalue. To prove the claim, it is enough to show that in the degree sequence $D(\mathcal{G}) = (\delta_1^{n_1}, \delta_2^{n_2}, \ldots, \delta_s^{n_s})$ there are distinct $k, \ell \in \{1, 2, \ldots, s\}$ such that $n_k, n_\ell > 1$, for then by Theorem 2.1 we have that either $m_k > 1$ or $m_\ell > 1$, that is, the eigenvalue μ_k or μ_ℓ is repeated. Note that, for any threshold graph $\mathcal{G}$, $d_1 = d_2$; hence in the sequence $D(\mathcal{G})$, there exists $k_* \in \{1, 2, \ldots, s\}$ such that $n_{k_*} > 1$.

Suppose that (a) holds for some $i \geq 2$. Then clearly $d_i = d_{i+1}$ and thus there exists an index $\ell \neq k_*$ such that $n_\ell > 1$. Hence, by Theorem 2.1, regardless of the fact that h is odd or even, either μ_ℓ or μ_{k_*} is a repeated eigenvalue of $\mathcal{G}$.

Now suppose that (b) holds. Then we have that

$$d_i = (i-1) + 1 + \sum_{j=i+2}^{n} T_{\mathcal{G}}(j),$$

$$d_{i+1} = i + \sum_{j=i+2}^{n} T_{\mathcal{G}}(j),$$

since v_i is adjacent to all vertices in $\{v_1, \dots, v_{i-1}\}$, to v_{i+1}, and to any vertex in $\{v_{i+2}, \dots, v_n\}$ that has a corresponding 1 in $T_{\mathcal{G}}$, and similar reasoning holds for computing d_{i+1}. As a result, $d_i = d_{i+1}$, and a similar argument as in case (a) establishes that $\mathcal{G}$ has a repeated eigenvalue.

Clearly, the binary creation sequences $T_{\mathcal{G}} = (0,1,0,1,0,1,\dots,0,1)$ and $T_{\mathcal{G}} = (0,0,1,0,1,0,1,\dots,0,1)$ are the only sequences that do not satisfy (a) and (b). Hence, it is enough to prove that the threshold graphs associated with these sequences have distinct eigenvalues. Consider first, $T_{\mathcal{G}} = (0,1,0,1,0,1,\dots,0,1)$, of even length n. It is straightforward to verify that the degree sequence is

$$D(\mathcal{G}) = \left(1, 2, \dots, \frac{n}{2} - 1, \frac{n}{2}, \frac{n}{2}, \frac{n}{2} + 1, \dots, n-1\right)$$

and therefore there are $h = n-1$ distinct degrees. From Theorem 2.1, the number of non-zero distinct eigenvalues is $n-1$, and hence all the eigenvalues of $T_{\mathcal{G}} = (0,1,\dots,0,1)$ are distinct. From Theorem 2.1 with $h = 2r+1$ odd, where $r = \frac{n}{2} - 1$, we have

$$\Lambda(\mathcal{G}) = \left(1, 2, \dots, \frac{n}{2} - 1, \frac{n}{2} + 1, \dots, n\right).$$

Finally, consider $T_{\mathcal{G}} = (0,0,1,0,1,0,1,\dots,0,1)$ of odd length n. In this case the degree sequence is

$$D(\mathcal{G}) = \left(1, 2, \dots, \frac{n-1}{2} - 1, \frac{n-1}{2}, \frac{n-1}{2}, \frac{n-1}{2} + 1, \dots, n-1\right)$$

and therefore there are $h = n-1$ distinct degrees and the same number of distinct non-zero eigenvalues. From Theorem 2.1 with $h = 2r$ odd, where $r = \frac{n-1}{2}$, we have

$$\Lambda(\mathcal{G}) = \left(1, 2, \dots, \frac{n-1}{2}, \frac{n-1}{2} + 2, \dots, n\right).$$

This completes the proof. □

Next, we construct the set of eigenvectors of the two classes of threshold graphs given in Proposition 3.2. To that end, it is straightforward to verify that for $T_{\mathcal{G}} = (0, 1, 0, 1, \ldots, 0, 1)$ of even length n we have

$$\mathbf{L} = \begin{bmatrix} d_1 & -1 & 0 & -1 & 0 & -1 & \cdots & 0 & -1 \\ -1 & d_2 & 0 & -1 & 0 & -1 & \cdots & 0 & -1 \\ 0 & 0 & d_3 & -1 & 0 & -1 & \cdots & 0 & -1 \\ -1 & -1 & -1 & d_4 & 0 & -1 & \cdots & 0 & -1 \\ 0 & 0 & 0 & 0 & d_5 & -1 & \cdots & 0 & -1 \\ -1 & -1 & -1 & -1 & -1 & d_6 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & d_{n-1} & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & \cdots & -1 & d_n \end{bmatrix}$$

and the degree sequence $d = (d_1, d_2, \ldots, d_n)$ is

$$d_j = \frac{n-j}{2} + (j-1) = \frac{n+j}{2} - 1$$

if j is even, and

$$d_i = \frac{n-(i+1)}{2} + 1 = \frac{n-i+1}{2}$$

if i is odd.

For each $\ell \in \{2, 3, \ldots, n\}$, let $\mathbf{u}_\ell = -\sum_{i=1}^{\ell-1} \mathbf{e}_i + (\ell-1)\mathbf{e}_\ell \in \mathbb{R}^n$ be the vector whose entries from 1 through $(\ell-1)$ are all equal to -1 and the ℓ-th entry is equal to $(\ell-1)$, and all other entries are equal to zero, that is,

$$\mathbf{u}_\ell = \begin{bmatrix} -1 & -1 & \cdots & -1 & (\ell-1) & 0 & \cdots & 0 \end{bmatrix}^T. \tag{2}$$

For example, for $n = 8$, if $\mathbf{U} = [\mathbf{1}_8 \; \mathbf{u}_2 \; \mathbf{u}_3 \; \cdots \; \mathbf{u}_8]$ then

$$\mathbf{U} = \begin{bmatrix} 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 2 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 3 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 4 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 5 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 6 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix}$$

Clearly, $\mathbf{1}_n^T \cdot \mathbf{u}_\ell = 0$ for $\ell \in \{2, 3, \ldots, n\}$.

Lemma 3.2. *Let $\mathcal{G}$ be the threshold graph with creation sequence*

$$T_{\mathcal{G}} = (0, 1, 0, 1, \dots, 0, 1)$$

of even order n. Let $\mathbf{u}_\ell$ be defined as in (2) for $\ell \in \{2, 3, \dots, n\}$. Then if $j \in \{2, 3, \dots, n\}$ is even we have

$$\mathbf{L}\mathbf{u}_j = (d_j + 1)\mathbf{u}_j, \tag{3}$$

and if $i \in \{2, 3, \dots, n\}$ is odd we have

$$\mathbf{L}\mathbf{u}_i = d_i \mathbf{u}_i. \tag{4}$$

In particular, $\{\mathbf{1}_n, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is a set of mutually orthogonal eigenvectors of $\mathbf{L}$.

Proof. It is clear that for $\mathbf{u}_\ell \in \{\mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ we have that $(\mathbf{L}\mathbf{u}_\ell)_a = 0$ for $a > \ell$, and thus we need only consider $(\mathbf{L}\mathbf{u})_a$ when $1 \le a \le \ell$.

Suppose first that $j \in \{2, 3, \dots, n\}$ is even. We consider three cases:

(i) Suppose that $a < j$ is even. Then, seeing as how the a-th row of $\mathbf{L}$ is

$$\begin{bmatrix} -1 & \cdots & -1 & d_a & 0 & -1 \cdots & 0 & -1 \end{bmatrix}$$

and that $d_a = \frac{n-a}{2} + (a - 1)$, a direct computation shows that

$$(\mathbf{L}\mathbf{u}_j)_a = -(d_j + 1).$$

(ii) Suppose that $a < j$ is odd. Then, seeing as how the a-th row of $\mathbf{L}$ is

$$\begin{bmatrix} 0 & \cdots & 0 & d_a & -1 & 0 & -1 \cdots & 0 & -1 \end{bmatrix}$$

and that $d_a = \frac{n-(a+1)}{2} + 1$, a direct computation shows that

$$(\mathbf{L}\mathbf{u}_j)_a = -(d_j + 1).$$

(iii) It is clear that for $a = j$, we have

$$(\mathbf{L}\mathbf{u}_j)_j = (j - 1) + (j - 1)d_j = (d_j + 1)(j - 1).$$

This proves the even case.

Suppose now that $i \in \{2, 3, \dots, n\}$ is odd. As in the even case, we consider three cases for a:

(i) If $a < i$ is even then a direct computation shows that

$$(\mathbf{Lu}_i)_a = -d_i.$$

(ii) If $a < i$ is odd then a direct computation shows that

$$(\mathbf{Lu}_i)_a = -d_i.$$

(iii) It is clear that for $a = i$ we have

$$(\mathbf{Lu}_i)_i = d_i(i-1).$$

This proves the odd case. □

Now consider the threshold graph $\mathcal{G}$ with creation sequence

$$T_{\mathcal{G}} = (0, 0, 1, 0, 1, \ldots, 0, 1)$$

of odd length n. It is straightforward to verify that

$$\mathbf{L} = \begin{bmatrix} d_1 & 0 & -1 & 0 & -1 & \cdots & 0 & -1 \\ 0 & d_2 & -1 & 0 & -1 & \cdots & 0 & -1 \\ -1 & -1 & d_3 & 0 & -1 & \cdots & 0 & -1 \\ 0 & 0 & 0 & d_4 & -1 & \cdots & 0 & -1 \\ -1 & -1 & -1 & -1 & d_5 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & d_{n-1} & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & \cdots & d_n \end{bmatrix}$$

and the degree sequence $d = (d_1, d_2, \ldots, d_n)$ is

$$d_j = \frac{n-j}{2} + (j-1) = \frac{n-(j-1)}{2} = \frac{n-j+1}{2},$$

if j is even, and

$$d_i = (i-1) + \frac{n-i}{2} = \frac{n+i}{2} - 1,$$

if i is odd. The proof of the following is very similar to that of Lemma 3.2 and is omitted.

Lemma 3.3. *Let $\mathcal{G}$ be the threshold graph with creation sequence*

$$T_{\mathcal{G}} = (0, 0, 1, 0, 1, \ldots, 0, 1)$$

of odd order n. Let $\mathbf{u}_\ell$ be defined as in (2) for $\ell \in \{2,3,\dots,n\}$. Then if $j \in \{2,3,\dots,n\}$ is even we have

$$\mathbf{L}\mathbf{u}_j = d_j \mathbf{u}_j$$

and if $i \in \{2,3,\dots,n\}$ is odd we have

$$\mathbf{L}\mathbf{u}_i = (d_i + 1)\mathbf{u}_i.$$

In particular, $\{\mathbf{1}_n, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is a set of mutually orthogonal eigenvectors of $\mathbf{L}$.

The complete classification of the eigenvectors of the threshold graphs $T_{\mathcal{G}} = (0,1,0,1,\dots,0,1)$ and $T_{\mathcal{G}} = (0,0,1,0,1,\dots,0,1)$ given in Lemma 3.2 and Lemma 3.3, respectively, allows us to completely classify the binary vectors $\mathbf{b}$ rendering $(\mathbf{L},\mathbf{b})$ controllable for threshold graphs. Before we prove our next main result, we need the following binary invariance controllability property [1].

Proposition 3.3. *Let $\mathcal{G}$ be a connected graph of order $n \geq 2$, let $\mathbf{b} \in \{0,1\}^n$, and let $\bar{\mathbf{b}} = \mathbf{1}_n - \mathbf{b}$ be the binary complement of $\mathbf{b}$. Then the pair $(\mathbf{L},\mathbf{b})$ is controllable if and only if the pair $(\mathbf{L},\bar{\mathbf{b}})$ is controllable. In fact,*

$$\dim\langle \mathbf{L}; \mathbf{b}\rangle = \dim\langle \mathbf{L}; \bar{\mathbf{b}}\rangle,$$

provided $\mathbf{b} \notin \{\mathbf{1}_n, \mathbf{0}_n\}$.

Theorem 3.3. *Let $\mathcal{G}$ be the threshold graph defined by $T_{\mathcal{G}} = (0,1,0,1,\dots,0,1)$ or $T_{\mathcal{G}} = (0,0,1,0,1,\dots,0,1)$ of order $n > 2$. Define the partition*

$$\{0,1\}^n = B_1 \cup B_2 \cup B_3 \cup B_4$$

as follows:

$$B_1 = \{\mathbf{b} \in \{0,1\}^n \mid (\mathbf{b})_1 = 1, (\mathbf{b})_2 = 0\}, \qquad B_3 = \{\mathbf{b} \in \{0,1\}^n \mid (\mathbf{b})_1 = (\mathbf{b})_2 = 0\},$$
$$B_2 = \{\mathbf{b} \in \{0,1\}^n \mid (\mathbf{b})_1 = 0, (\mathbf{b})_2 = 1\}, \qquad B_4 = \{\mathbf{b} \in \{0,1\}^n \mid (\mathbf{b})_1 = (\mathbf{b})_2 = 1\}.$$

The following hold:

(i) *The pair $(\mathbf{L},\mathbf{b})$ is controllable if and only if $\mathbf{b} \in B_1 \cup B_2$.*
(ii) *Exactly half of the binary vectors $\mathbf{b} \in \{0,1\}^n$ yield a controllable pair $(\mathbf{L},\mathbf{b})$.*
(iii) *If $\mathbf{b} \in B_3 \backslash \{\mathbf{0}_n\}$ and*

$$\ell_1(\mathbf{b}) := \min\{\ell \in \{3,\dots,n\} \mid (\mathbf{b})_\ell = 1\},$$

then $\dim\langle \mathbf{L}; \mathbf{b}\rangle = n + 2 - \ell_1(\mathbf{b})$.

(iv) *If* $\mathbf{b} \in B_4 \backslash \{\mathbf{1}_n\}$ *and*

$$\ell_0(\mathbf{b}) := \min\bigl\{\ell \in \{3, \dots, n\} \bigm| (\mathbf{b})_\ell = 0\bigr\},$$

then $\dim\langle \mathbf{L}; \mathbf{b}\rangle = n + 2 - \ell_0(\mathbf{b})$.

Proof. To prove (i) let $\mathbf{b} \in B_1$. Then clearly $\mathbf{u}^T \cdot \mathbf{b} \neq 0$ for every eigenvector $\mathbf{u} \in \{\mathbf{1}_n, \mathbf{u}_2, \dots, \mathbf{u}_n\}$, and therefore $(\mathbf{L}, \mathbf{b})$ is controllable. Hence, every vector $\mathbf{b} \in B_1$ yields a controllable pair $(\mathbf{L}, \mathbf{b})$. Now let $\mathbf{b} \in B_2$. Then clearly $\bar{\mathbf{b}} \in B_1$, and therefore by invariance of controllability under binary complements, it follows that $(\mathbf{L}, \mathbf{b})$ is also controllable for every $\mathbf{b} \in B_2$. Now let $\mathbf{b} \in B_3$ and consider the eigenvector $\mathbf{u}_2 = [1\ {-1}\ 0\ 0\ \cdots\ 0]^T$. Then clearly $\mathbf{u}_2^T \cdot \mathbf{b} = 0$, and therefore $(\mathbf{L}, \mathbf{b})$ is uncontrollable. If now $\mathbf{b} \in B_4$ then clearly also $\mathbf{u}_2^T \cdot \mathbf{b} = 0$, and thus $(\mathbf{L}, \mathbf{b})$ is uncontrollable.

Part (ii) follows from $|B_1| + |B_2| = 2^{n-2} + 2^{n-2} = 2^{n-1}$.

To prove (iii), let $\mathbf{b} \in B_3 \backslash \{\mathbf{0}_n\}$ and let $\ell_1 = \ell_1(\mathbf{b})$ be as above. Then by definition of $\mathbf{u}_\ell$ in (2), we have that $\mathbf{u}_2^T \cdot \mathbf{b} = \cdots = \mathbf{u}_{\ell_1 - 1}^T \cdot \mathbf{b} = 0$ and $\mathbf{u}_\ell^T \cdot \mathbf{b} \neq 0$ for $\ell \in \{\ell_1, \ell_1 + 1, \dots, n\}$. Also, it is clear that $\mathbf{1}_n^T \cdot \mathbf{b} \neq 0$. Hence, the number of eigenvectors that $\mathbf{b}$ is not orthogonal to is $n - (\ell_1 - 1) - 1 = n - \ell_1 + 2$. The result now follows since $\dim\langle \mathbf{L}; \mathbf{b}\rangle$ is equal to the number of eigenvectors that $\mathbf{b}$ is not orthogonal to.

To prove (iv), if $\mathbf{b} \in B_4$ then clearly $\ell_0(\mathbf{b}) = \ell_1(\bar{\mathbf{b}})$. Then, by invariance of controllability under binary complements,

$$\dim\langle \mathbf{L}; \mathbf{b}\rangle = \dim\langle \mathbf{L}; \bar{\mathbf{b}}\rangle = n + 2 - \ell_1(\bar{\mathbf{b}}) = n + 2 - \ell_0(\bar{\mathbf{b}}).$$

This ends the proof. □

We now prove Theorem 3.2.

Proof of Theorem 3.2. If $\mathbf{L}$ has a repeated eigenvalue then $(\mathbf{L}, \mathbf{b})$ is uncontrollable for every $\mathbf{b} \in \mathbb{R}^n$, and in particular for every $\mathbf{b} \in \{0, 1\}^n$.

Now suppose that $\mathcal{G}$ is completely uncontrollable. Assume by contradiction that $\mathcal{G}$ has simple eigenvalues. Then $\mathcal{G}$ is either the threshold graph $T_\mathcal{G} = (0, 1, 0, 1, \dots, 0, 1)$ or $T_\mathcal{G} = (0, 0, 1, 0, 1, \dots, 0, 1)$. By Theorem 3.3, $\mathcal{G}$ is conditionally controllable, which is a contradiction. This completes the proof. □

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