# Math 333 - Practice Exam with *Some* Solutions

(Note that the exam will NOT be this long.)

## 1 Definitions

**1. (0 points)** Let $U$ be a subset of a vector space $V$. Let $S = \{v_1, v_2, \ldots, v_n\}$ be another subset of $V$.
(a) Define "$U$ is a **subspace** of $V$".

(b) Define "$S$ is **linearly independent**".

(c) Define "$S$ **generates** $V$".

## 2 Vector Spaces and Subspaces

**2. (0 points)**
(a) Give three examples of 4-dimensional vector spaces.

(b) Give one example of an infinite dimensional vector space.

(c) Give an example of a zero-dimensional vector space.

**3. (0 points)** Let $S_1$ and $S_2$ be subspaces of a vector space $V$. Prove that the union $S_1 \cup S_2$ is a subspace of $V$ if and only if one is contained in the other (that is, either $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$.)

**Solution:** ($\Longleftarrow$) $S_1$ and $S_2$ are subspaces. If $S_1 \subseteq S_2$, then $S_1 \cup S_2 = S_2$ is a subspace. If $S_2 \subseteq S_1$, then $S_1 \cup S_2 = S_1$ is a subspace. We've proved one direction.

($\Longrightarrow$) $S_1$ and $S_2$ are subspaces, and suppose $S_1 \cup S_2$ is a subspace. If $S_1 \subseteq S_2$, then we are done. If $S_1 \nsubseteq S_2$, then we *need to show* $S_2 \subseteq S_1$.
Choose $x \in S_2$. Since $S_1 \nsubseteq S_2$ there must be some vector in $S_1$ that is not in $S_2$, call it $y$. So $y \in S_1$, but $y \notin S_2$. Since $S_1 \cup S_2$ is a subspace, it is closed under addition and

$x + y$ must be in $S_1 \cup S_2$ since $x \in S_2 \subseteq S_1 \cup S_2$ and $y \in S_1 \subseteq S_1 \cup S_2$. Thus we must have either $x + y \in S_1$ or $x + y \in S_2$.
If $x + y \in S_2$, then since $x \in S_2$ and $S_2$ is a subspace (i.e. closed under the operations) we have $y = (x + y) - x \in S_2$, which contradicts the fact that $y \notin S_2$. Thus $x + y \in S_1$. However, since $y \in S_1$ and $S_1$ is a subspace (i.e. closed under the operations) we have $x = (x + y) - y \in S_1$. Therefore, $S_2 \subseteq S_1$.

# 3 Linear Independence, Generating Sets, and Bases

**4. (0 points)** Let $S = \{x^2 + 3x, x - 2\}$ be a subset of $P_2(\mathbb{R})$.
(a) Explain why $S$ is *not* a basis of $P_2(\mathbb{R})$.

(b) Is $\frac{1}{3}x^2 + 2$ in span$(S)$? Explain.

(c) Is $2x^2 + 5x + 4$ in span$(S)$? Explain.

**5. (0 points)** Consider the 3 vectors in $\mathbb{R}^3$ given by $v_1 = (1, 1, -1)$, $v_2 = (1, 1, 1)$, and $v_3 = (3, 5, 7)$. Decide whether these 3 vectors provide a basis for $\mathbb{R}^3$. Justify your answer.

**6. (0 points)** Let $W$ be the subspace of $\mathbb{R}^3$ given by

$$W = \{(x, y, z) \mid x + y + z = 0 \text{ and } x - y - z = 0\}.$$

Find a basis for $W$ and the dimension of $W$.

**7. (0 points)** Let $S = \{v_1, v_2, \ldots, v_n\}$ be a set of $n$ vectors in a vector space $V$. Show that if $S$ is linearly independent and the dimension of $V$ is $n$, then $S$ is a basis of $V$.

**Solution:** This is Corollary 2 (b) at the top of page 48 of the textbook. The proof is found there.

**8. (0 points)** Consider the subset $S = \{x^3 - 2x^2 + 1, 4x^2 - x + 3, 3x - 2\}$ of $P_3(\mathbb{R})$.

(a) Explain how you know that $S$ does not generate $P_3(\mathbb{R})$.

**Solution:** Since $S$ has 3 vectors and the dimension of $P_3(\mathbb{R})$ is 4, $S$ cannot generate $P_3(\mathbb{R})$.

(b) Can you add a vector $v$ to $S$ so that $S \cup \{v\}$ is a basis of $P_3(\mathbb{R})$? Justify and find such a vector if possible.

**Solution:** As long as $S$ is linearly independent we know that $S$ can be extended to a basis. To see $S$ is linearly independent suppose that

$$a(x^3 - 2x^2 + 1) + b(4x^2 - x + 3) + c(3x - 2) = 0.$$

This clearly implies that $a = 0$ since only one term has an $x^3$. So now

$$b(4x^2 - x + 3) + c(3x - 2) = 0,$$

and again we see that $b = 0$. Clearly $c$ must also be 0. Furthermore, we can add $v = 1$ as the last vector using a similar argument to show this new set is linearly independent. Then since the dimension of $P_3(\mathbb{R})$ is 4, we know this new set is a basis.

**9. (0 points)** Let $V$ be a vector space over $\mathbb{R}$, and let $x, y, z \in V$. Prove that $\{x, y, z\}$ is linearly independent if and only if $\{x + y, y + z, z + x\}$ is linearly independent.

**Solution:** ($\Longrightarrow$) Assume that $\{x, y, z\}$ is linearly independent. Suppose there are $a, b, c \in \mathbb{R}$ such that

$$a(x + y) + b(y + z) + c(z + x) = 0.$$

So $0 = a(x + y) + b(y + z) + c(z + x) = (a + c)x + (a + b)y + (b + c)z$, and this means that $a + c = a + b = b + c = 0$ since $\{x, y, z\}$ is linearly independent. Clearly from those equalities we have $a = b = c = 0$. Therefore, $\{x + y, y + z, z + x\}$ is also linearly independent.

($\Longleftarrow$) Assume that $\{x + y, y + z, z + x\}$ is linearly independent. Suppose there are $a, b, c \in \mathbb{R}$ such that

$$ax + by + cz = 0.$$

So $0 = ax + by + cz = (\frac{a+b-c}{2})(x+y) + (\frac{b+c-a}{2})(y+z) + (\frac{c+a-b}{2})(z+x)$, and this means that $a+b-c = b+c-a = c+a-b = 0$ since $\{x+y, y+z, z+x\}$ is linearly independent. Clearly from those equalities we have $a = b = c = 0$. Therefore, $\{x, y, z\}$ is also linearly independent.

**10. (0 points)** Let $S_1$ and $S_2$ be subsets of a vector space $V$ over a field $F$. Prove that

$$\operatorname{span}(S_1 \cap S_2) \subseteq \operatorname{span}(S_1) \cap \operatorname{span}(S_2).$$

**Solution:** Let $x \in \operatorname{span}(S_1 \cap S_2)$. Then there exist vectors $v_1, v_2, \ldots, v_n \in S_1 \cap S_2$ and coefficients $a_1, a_2, \ldots, a_n \in F$ such that $x = a_1v_1 + a_2v_2 + \cdots + a_nv_n$. But since $v_1, v_2, \ldots, v_n \in S_1$, we see that $x \in \operatorname{span}(S_1)$. Similarly, since $v_1, v_2, \ldots, v_n \in S_2$, we see that $x \in \operatorname{span}(S_2)$. Thus we have $x \in \operatorname{span}(S_1) \cap \operatorname{span}(S_2)$.

**11. (0 points)** Consider the vector space $V = P_1(\mathbb{R})$.

(a) Explain why you know that the set $\beta = \{1+x, 1-2x\}$ is a basis of $V$.

**Solution:** Since neither vector is a multiple of the other, $\beta$ is linearly independent. Since the dimension of $V$ is 2 and $\beta$ has 2 elements, it must be a basis.

(b) Express $p(x) = 2x - 3$ as a linear combination of $\beta$.

**Solution:** $p(x) = 2x - 3 = (-4/3)(1+x) + (-5/3)(1-2x)$