Math 333 - Practice Exam 2

(Note that the exam will NOT be this long.)

1 Definitions

1. (0 points) Let $T : V \to W$ be a transformation. Let A be a square matrix.

(a) Define "T is linear".

(b) Define the null space of T, null(T).

(c) Define the image of T, image(T).

(d) Define "T is one-to-one".

(e) Define "T is onto".

(f) Define "T is invertible".

(g) Define "T is an isomorphism".

(h) Define rank(T) and nullity(T).

(i) Define "A is invertible".

2 Linear Transformations, Null Spaces, and Images

2. (0 points) Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be given by $T(f(x)) = f(x) - xf'(x)$.

(a) Show T is linear.

(b) Find a basis for the image of T.

(c) Is T one-to-one? Is T onto? Justify your answer.

3. (0 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(x, y, z) = (x + y, x - z, 2x + 3y + z) \ .$$

(a) Show T is linear.

(b) Find a basis for null(T).

(c) Find a basis for image(T).

(d) State the Dimension Theorem and verify that T satisfies it.

(e) Is T one-to-one? Onto? Explain.

4. (0 points) Let V and W be finite-dimensional vector spaces and $T : V \to W$ be linear.

(a) Prove that if $\dim(V) < \dim(W)$, then T cannot be onto.

(b) Prove that if $\dim(V) > \dim(W)$, then T cannot be one-to-one.

5. (0 points) Let $T : V \longrightarrow W$ be a linear transformation. Prove the following theorems.

(a) **Theorem 2.1:** The sets null(T) and image(T) are subspaces of V and W, respectively.

(b) **Theorem 2.2:** Let β be a basis of V. Then the set $\{T(\beta)\}$ is a generating set for image(T).

(c) **Theorem 2.4:** T is one-to-one if and only if null(T) $= \{0\}$.

3 Matrix Representations and Change of Basis

6. (0 points) Consider the vector space $V = P_1(\mathbb{R})$.

(a) Explain why you know that the set $\beta = \{1 + x, 1 - 2x\}$ is a basis of V.

(b) Determine $[p(x)]_\beta$, where $p(x) = 2x - 3 \in V$.

7. (0 points) Let $T : P_2(\mathbb{R}) \longrightarrow \mathbb{R}^2$ be given by $T(f(x)) = (f(0), f'(1))$.

(a) Show that T is linear.

(b) Determine the matrix of T with respect to the standard bases of $P_2(\mathbb{R})$ and $\mathbb{R}^2$.

8. (0 points) Let β and γ be the following standard ordered bases of $M_{2\times 2}(\mathbb{R})$ and $P_2(\mathbb{R})$, respectively:

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \text{ and } \gamma = \{1, x, x^2\}.$$

Compute $[T]_\gamma^\beta$ if we define the linear transformation $T : P_2(\mathbb{R}) \longrightarrow M_{2\times 2}(\mathbb{R})$ by

$$T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}.$$

9. (0 points) Let V, W, and Z be vector spaces, and let $T : V \to W$ and $U : W \to Z$ be linear transformations.

(a) Prove that if $U \circ T$ is one-to-one, then T is one-to-one.

(b) Prove that if $U \circ T$ is onto, then U is onto.

(c) Prove that if U and T are one-to-one and onto, then $U \circ T$ is also

10. (0 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(x, y, z) = (x + y + z, x + 3y + 5z)$$

Let β and γ be the standard bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively. Also consider another basis $\alpha = \{(1, 1, 1), (2, 3, 4), (3, 4, 6)\}$ for $\mathbb{R}^3$.

(a) Compute the matrix representation $[T]_\beta^\gamma$.

(b) Compute the matrix representation $[T]_\alpha^\gamma$.

(c) Compute Q the change of coordinate matrix from β to α.

(d) Check that $[T]_\alpha^\gamma \cdot Q = [T]_\beta^\gamma$.

(e) Let $x = (1, 5, 7)$. What is $[x]_\beta$? Use this, together with Q, to find $[x]_\alpha$.