Math 333 - Practice Exam 2 with *Some* Solutions

(Note that the exam will NOT be this long.)

1 Definitions

1. (0 points) Let $T : V \to W$ be a transformation. Let A be a square matrix.

(a) Define "T is linear".

(b) Define the null space of T, null(T).

(c) Define the image of T, image(T).

(d) Define "T is one-to-one".

(e) Define "T is onto".

(f) Define "T is invertible".

(g) Define "T is an isomorphism".

(h) Define rank(T) and nullity(T).

(i) Define "A is invertible".

Solution: See your notes or textbook.

2 Linear Transformations, Null Spaces, and Images

2. (0 points) Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be given by $T(f(x)) = f(x) - xf'(x)$.

(a) Show T is linear.

Solution: Let $a \in \mathbb{R}$ and $f(x), g(x) \in P_2(\mathbb{R})$. Then

$$\begin{aligned} T(af(x)+g(x)) &= [af(x)+g(x)] - x[af(x)+g(x)]' \\ &= af(x)+g(x) - axf'(x) - xg'(x) \\ &= a[f(x) - xf'(x)] + [g(x) - xg'(x)] \\ &= aT(f(x)) + T(g(x)). \end{aligned}$$

(b) Find a basis for the image of T.

Solution: We know that a generating set for the image of T is the image of the standard basis of $P_2(\mathbb{R})$. Thus

$$\begin{aligned} \text{image}(T) &= \text{span}(\{T(1), T(x), T(x^2)\}) \\ &= \text{span}(\{1, x - x, x^2 - 2x^2\}) \\ &= \text{span}(\{1, -x^2\}). \end{aligned}$$

The vectors $\{1, -x^2\}$ are clearly linearly independent, so it will also be a basis.

(c) Is T one-to-one? Is T onto? Justify your answer.

Solution: Since $\text{rank}(T) = 2$ and $\dim(P_2(\mathbb{R})) = 3$, T is clearly not onto. Furthermore, the Dimension Theorem says the $\text{nullity}(T) = 1$, so T is not one-to-one either.

3. (0 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(x, y, z) = (x + y, x - z, 2x + 3y + z) \ .$$

(a) Show T is linear.

(b) Find a basis for null(T).

(c) Find a basis for image(T).

(d) State the Dimension Theorem and verify that T satisfies it.

(e) Is T one-to-one? Onto? Explain.

4. (0 points) Let V and W be finite-dimensional vector spaces and $T : V \to W$ be linear.

(a) Prove that if $\dim(V) < \dim(W)$, then T cannot be onto.

Solution: Suppose $\dim(V) < \dim(W)$, and assume (by means of contradiction) that T is onto. Then image(T) $= W$, and thus rank(T) $= \dim(W)$. By the dimensions theorem, we know

$$\begin{aligned} \dim(V) &= \text{rank}(T) + \text{nullity}(T) \\ &= \dim(W) + \text{nullity}(T) \end{aligned}$$

Since $\dim(V) < \dim(W)$, this implies $\text{nullity}(T) = \dim(V) - \dim(W) < 0$, which is a contradiction since nullity can not be negative. Thus T is NOT onto.

(b) Prove that if $\dim(V) > \dim(W)$, then T cannot be one-to-one.

Solution: Similar argument to (a). See if you can get it.

5. (0 points) Let $T : V \longrightarrow W$ be a linear transformation. Prove the following theorems.

(a) **Theorem 2.1:** The sets null(T) and image(T) are subspaces of V and W, respectively.

(b) **Theorem 2.2:** Let β be a basis of V. Then the set $\{T(\beta)\}$ is a generating set for image(T).

(c) **Theorem 2.4:** T is one-to-one if and only if null$(T) = \{0\}$.

Solution: See your notes or textbook.

3 Matrix Representations and Change of Basis

6. (0 points) Consider the vector space $V = P_1(\mathbb{R})$.

(a) Explain why you know that the set $\beta = \{1 + x, 1 - 2x\}$ is a basis of V.

Solution: Since neither vector is a multiple of the other, β is linearly independent. Since the dimension of V is 2, β is a basis.

(b) Determine $[p(x)]_\beta$, where $p(x) = 2x - 3 \in V$.

Solution: Notice that $p(x) = 2x - 3 = (-4/3)(1 + x) + (-5/3)(1 - 2x)$. Therefore $[p(x)]_\beta = \begin{pmatrix} -4/3 \\ -5/3 \end{pmatrix}$.

7. (0 points) Let $T : P_2(\mathbb{R}) \longrightarrow \mathbb{R}^2$ be given by $T(f(x)) = (f(0), f'(1))$.

(a) Show that T is linear.

Solution: Let $f(x), g(x) \in P_2(\mathbb{R})$ and $c \in \mathbb{R}$. Then

$$T(cf(x) + g(x)) = (cf(0) + g(0), cf'(1) + g'(1))$$

$$= c(f(0), f'(1)) + (g(0), g'(1))$$

$$= cT(f(x)) + T(g(x)).$$

(b) Determine the matrix of T with respect to the standard bases of $P_2(\mathbb{R})$ and $\mathbb{R}^2$.

Solution: First we recall that the standard basis of $P_2(\mathbb{R})$ is $\beta = \{1, x, x^2\}$ and that the standard basis of $\mathbb{R}^2$ is $\gamma = \{(1,0),(0,1)\}$. Now we look at the image of each element of the basis β under T.

$$T(1) = (1,0),\ T(x) = (0,1),\ \text{and}\ T(x^2) = (0,2).$$

Since we are using the standard basis of $\mathbb{R}^2$ the columns of our matrix are the vectors we have just written. So our matrix is

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

8. (0 points) Let β and γ be the following standard ordered bases of $M_{2\times 2}(\mathbb{R})$ and $P_2(\mathbb{R})$, respectively:

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \text{ and } \gamma = \{1, x, x^2\}.$$

Compute $[T]_\gamma^\beta$ if we define the linear transformation $T : P_2(\mathbb{R}) \longrightarrow M_{2\times 2}(\mathbb{R})$ by

$$T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}.$$

Solution: First we see that $T(1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$. So the first column of $[T]_\gamma^\beta$ is the coordinate vector $[T(1)]_\beta = (0,2,0,0)$. Next $T(x) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$. So the second column of $[T]_\gamma^\beta$ is the coordinate vector $[T(x)]_\beta = (1,2,0,0)$. Finally $T(x^2) = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$. So the third column of $[T]_\gamma^\beta$ is the coordinate vector $[T(x^2)]_\beta = (0,2,0,2)$. So in total we get

$$[T]_\gamma^\beta = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

9. (0 points) Let V, W, and Z be vector spaces, and let $T : V \to W$ and $U : W \to Z$ be linear transformations.

(a) Prove that if $U \circ T$ is one-to-one, then T is one-to-one.

Solution: Suppose $U \circ T$ is one-to-one. Then $\text{null}(U \circ T) = \{0\}$. What is $\text{null}(T)$?

Suppose $x \in \text{null}(T)$, then $T(x) = 0$ and $(U \circ T)(x) = U(T(x)) = U(0) = 0$. So x is in $\text{null}(U \circ T)$ also. But 0 is the only thing in $\text{null}(U \circ T)$, so $x = 0$, and we have shown that $\text{null}(T) = \{0\}$. Therefore T is one-to-one.

(b) Prove that if $U \circ T$ is onto, then U is onto.

Solution: Similar argument to (a). See if you can get it.

(c) Prove that if U and T are one-to-one and onto, then $U \circ T$ is also

Solution: Similar argument to (a). See if you can get it.

10. (0 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(x, y, z) = (x + y + z, x + 3y + 5z)$$

Let β and γ be the standard bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively. Also consider another basis $\alpha = \{(1, 1, 1), (2, 3, 4), (3, 4, 6)\}$ for $\mathbb{R}^3$.

(a) Compute the matrix representation $[T]_\beta^\gamma$.

(b) Compute the matrix representation $[T]_\alpha^\gamma$.

(c) Compute Q the change of coordinate matrix from β to α.

(d) Check that $[T]_\alpha^\gamma \cdot Q = [T]_\beta^\gamma$.

(e) Let $x = (1, 5, 7)$. What is $[x]_\beta$? Use this, together with Q, to find $[x]_\alpha$.

Solution:

(a) Plugging basis β into T and writing as a linear combination of the elements of γ, we get $[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix}$.

(b) Plugging basis α into T and writing as a linear combination of the elements of γ, we get $[T]_\alpha^\gamma = \begin{pmatrix} 3 & 9 & 13 \\ 9 & 31 & 45 \end{pmatrix}$.

(c) To get the change of basis matrix, we must find the coordinate vectors of the elements of β with respect to α:

$$[(1,0,0)]_\alpha = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \ [(0,1,0)]_\alpha = \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix}, \text{ and } [(0,0,1)]_\alpha = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

Therefore the change of basis matrix is $Q = \begin{pmatrix} 2 & 0 & -1 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix}$.

(d) $[T]_\alpha^\gamma \cdot Q = \begin{pmatrix} 3 & 9 & 13 \\ 9 & 31 & 45 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix} = [T]_\beta^\gamma$.

(e) $x = (1,5,7) \Longrightarrow [x]_\beta = \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}$

$$\Longrightarrow [x]_\alpha = Q \cdot [x]_\beta = \begin{pmatrix} 2 & 0 & -1 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} -5 \\ 6 \\ -2 \end{pmatrix}$$