Math 333 - Practice Final Exam with *Some* Solutions

(Note that the exam will NOT be this long.)

1 Definitions

1. (0 points) Let A be an $m \times n$ matrix and $b \in \mathbb{R}^m$. Let $T : V \longrightarrow V$ be a linear transformation on a vector space V.

(a) Define "$Ax = b$ is consistent".

(b) Define "$Ax = b$ is homogeneous".

(c) If $m = n$, then define "A is diagonalizable".

(d) Define "T is diagonalizable".

(e) If $m = n$, then define "v is an eigenvector of A with eigenvalue λ."

(f) Define eigenvalue and eigenvector of T.

(g) Define the eigenspace of A associated to eigenvalue λ.

2 Inverses

2. (0 points) Consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

(a) Find the inverse of A (using the Gaussian elimination method).

Solution: $A^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$

(b) Show that your matrix is, in fact, the inverse of A.

Solution: $AA^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and

$$A^{-1}A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3. (0 points) Derive the formula for the inverse of a 2×2 matrix. That is, what is A^{-1} if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$? (Don't just state the formula.)

Solution: Hint: The definition of inverse says A^{-1} is also a 2×2 matrix, say $A^{-1} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, such that $AA^{-1} = I$. Multiply A and A^{-1}, set it equal to I, and solve for x, y, z, and w in terms of a, b, c, and d.

4. (0 points) Consider the matrix $A = \begin{pmatrix} 1 & 1 & 3 \\ -1 & 0 & 1 \\ 2 & 1 & -1 \end{pmatrix}$.

(a) Find the inverse of A.

(b) Suppose A is the change of basis matrix from the standard ordered basis $\alpha = \{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ to some other ordered basis $\beta = \{b_1, b_2, b_3\}$ of $\mathbb{R}^3$. Then what is the ordered basis β?

Solution: (a) $A^{-1} = \begin{pmatrix} 1/3 & -4/3 & -1/3 \\ -1/3 & 7/3 & 4/3 \\ 1/3 & -1/3 & -1/3 \end{pmatrix}$.

(b) Suppose $\beta = \{b_1, b_2, b_3\}$ and $\alpha = \{e_1, e_2, e_3\}$. Since A is the change of basis matrix from α to β, then $A[b_i]_\alpha = [b_i]_\beta$. Since $[b_i]_\beta = e_i$ and $[b_i]_\alpha = b_i$, we get $Ab_i = e_i$, or equivalently, $A^{-1}e_i = b_i$. So the i^{th} vector in β is the i^{th} column of A^{-1}. Therefore,

$$\beta = \left\{ \begin{pmatrix} 1/3 \\ -1/3 \\ 1/3 \end{pmatrix}, \begin{pmatrix} -4/3 \\ 7/3 \\ -1/3 \end{pmatrix}, \begin{pmatrix} -1/3 \\ 4/3 \\ -1/3 \end{pmatrix} \right\}.$$

3 Systems of Linear Equations

5. (0 points) Consider the following system of linear equations.

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 2 \\ 2x_1 + x_2 + x_3 - x_4 &= 3 \\ x_1 + 2x_2 - 3x_3 + 2x_4 &= 2 \end{aligned}$$

(a) Rewrite the system as a matrix equation $Ax = b$.

(b) Solve the system using elementary row operations.

(c) Find the column space of A and null(A) by specifying a basis for each.

(d) Determine rank(A) and nullity(A).

Solution:

(a) $\begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 1 & 1 & -1 \\ 1 & 2 & -3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$

(b) Row reducing the augmented matrix gives

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 2 & 1 & 1 & -1 & 3 \\ 1 & 2 & -3 & 2 & 2 \end{array}\right) \dashrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1/2 & 4/3 \\ 0 & 1 & 0 & 1/2 & 1/3 \\ 0 & 0 & 1 & -1/2 & 0 \end{array}\right).$$

Back-solving tells us the solution set is $\left\{ \begin{pmatrix} 4/3 \\ 1/3 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 1 \end{pmatrix} \right\}$.

(c) The basis for the column space of A is the set of linearly independent columns of A. Since the reduced row echelon form of A is $\text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1/2 \end{pmatrix}$, we see that the first three columns all have pivots and the corresponding columns of A,

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} \right\},$$

form the basis of the column space. The nullspace of A has basis $\left\{ \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 1 \end{pmatrix} \right\}$.

(d) $\text{rank}(A) = \text{dimension of column space} = 3$ and $\text{nullity}(A) = \text{dimension of null}(A) = 1$.

6. (0 points) Consider the following inhomogeneous system of linear equations.

$$\begin{aligned} x + y + z + w &= 1 \\ x + y &= 0 \\ x + w &= 0 \end{aligned}$$

(a) Write the coefficient matrix A for this system.

(b) Find the solution space to the associated *homogeneous* system.

(c) By simple inspection, find *just one* solution to the given *inhomogeneous* system.

(d) Find all solutions to the given *inhomogeneous* system.

7. (0 points) Consider the matrix $A = \begin{pmatrix} 1 & 1 \\ -2 & h \end{pmatrix}$ and vector $b = \begin{pmatrix} k \\ 1 \end{pmatrix}$. Find all possible values of h and k so that the matrix equation $Ax = b$ has:

(a) no solution.

(b) exactly one solution.

(c) infinitely many solutions.

Solution: For each case we need to look at the augmented matrix $(A|b) = \begin{pmatrix} 1 & 1 & k \\ -2 & h & 1 \end{pmatrix}$.

Performing one row operation we arrive at the new augmented matrix

$$(B|c) = \begin{pmatrix} 1 & 1 & k \\ 0 & h+2 & 1+2k \end{pmatrix},$$

and the system $Ax = b$ is equivalent to $Bx = c$.

(a) $Bx = c$ has no solutions if and only if $\text{rank}(B) < \text{rank}(B|c)$. This is when the second column does not have a pivot,but the augmented column does. This occurs if and only if $h = -2$ and $k \neq -1/2$.

(b) $Bx = c$ has exactly one solution if and only if B is invertible which occurs if and only if $\text{rank}(B) = 2$ if and only if the first and second columns have pivots if and only if $h \neq -2$.

(c) $Bx = c$ has infinitely many solutions in the remaining cases and so if and only if $h = -2$ and $k = -1/2$.

8. (0 points) Suppose A is an $m \times n$ matrix with $\text{rank}(A) = 0$. Prove $A = O$.

Solution: The fact that $\text{rank}(A) = 0$ implies that there are NO pivot columns. So there are no pivots, which means NO row has a first non-zero entry. Thus each row consists entirely of zeros. Therefore $A = O$.

4 Determinants

9. (0 points) Find the determinants of the following matrices in two ways:

- By using cofactor expansion along a row or column.
- By invoking some basic facts about determinants. Please state which general facts about determinants you are invoking. (In other words, you must explain what you are doing.)

(a) $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(b) $B = \begin{pmatrix} 3 & 2 & 3 \\ 1 & 5 & 1 \\ 4 & 6 & 4 \end{pmatrix}$

(c) $C = \begin{pmatrix} 3 & 4 & 1 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 2 & 3 \\ 3 & 4 & 1 & 5 \end{pmatrix}$

Solution: (a) Switching rows one and three give you I_3. So $\det(A) = -\det(I_3) = -1$. Using cofactor expansion along the first row we get

$$\det(A) = 0 - 0 + (1)\det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1.$$

(b) Notice that the first and third columns are identical. So $\det(B) = 0$. Also do cofactor expansion.

(c) Subtracting row one from row four (which does not change the determinant) yields an upper triangular matrix with diagonal entries 3, 1, 2, and 5. So $\det(C) = 3 \cdot 1 \cdot 2 \cdot 5 = 30$. Cofactor expansion will lead to the same answer.

10. (0 points) Using row reduction show that:

$$\det\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} = (b-a)(c-a)(c-b)$$

Solution: Performing only type 3 row operations (adding a multiple of one row to another row) does not change the determinant and reduces the matrix to

$$\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c-a)(c-b) \end{pmatrix}.$$

The determinant is the product of the diagonal entries $(b-a)(c-a)(c-b)$.

5 Diagonalization and Eigenspaces

11. (0 points) Consider $A \in M_{n\times n}(F)$. Prove that the vector $v \in \mathbb{R}^n$ ($v \neq \vec{0}$) is an eigenvector of A corresponding to eigenvalue λ if and only if $v \in \text{null}\,(A - \lambda I)$.

Solution: The vector v is an eigenvector of A corresponding to eigenvalue λ

$$\begin{aligned} &\iff Av = \lambda v \\ &\iff Av - \lambda v = \vec{0} \\ &\iff (A - \lambda I)v = \vec{0} \\ &\iff v \in \text{null}\,(A - \lambda I) \end{aligned}$$

12. (0 points) Find all eigenvalues and the corresponding eigenspaces for the following matrices. For each eigenvalue, give the algebraic multiplicity and geometric multiplicity.

(a) $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

Solution: (a) $p(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 0 \\ 2 & 1-\lambda \end{pmatrix} = (1-\lambda)(1-\lambda) = \lambda^2 - 2\lambda + 1.$ So the eigenvalues are repeated with $\lambda = 1$. The eigenspace is

$$E_\lambda = \text{null}(A - I) = \text{null} \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} = \left\{ t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

The eigenvalue $\lambda = 1$ has algebraic multiplicity 2 and geometric multiplicity 1.

(b) $p(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = (1-\lambda)(1-\lambda) - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$. So the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -1$. The eigenspace for λ_1 is

$$E_{\lambda_1} = \text{null}(A - 3I) = \text{null} \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} = \text{null} \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} = \left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

The eigenspace for λ_2 is

$$E_{\lambda_2} = \text{null}(A + I) = \text{null} \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \text{null} \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} = \left\{ t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

The eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$ both have algebraic multiplicity 1 and geometric multiplicity 1.

13. (0 points) Consider the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

(a) Find the characteristic polynomial of A.

(b) A is diagonalizable. That is, $[L_A]_\beta$ is a diagonal matrix for some basis β of $\mathbb{R}^3$. Find such a basis β.

(c) For the basis β you found above, what is $[L_A]_\beta$?

(d) We also know that A is similar to a diagonal matrix D. Find the matrices D and Q such that $D = Q^{-1}AQ$.

Solution: The characteristic polynomial is

$$p(\lambda) = \det \begin{pmatrix} 1-\lambda & 0 & 0 \\ -1 & 1-\lambda & 1 \\ 1 & 0 & -\lambda \end{pmatrix} = -\lambda(1-\lambda)^2.$$

Thus the eigenvalues of A are 0 and 1. To get the basis β we must determine the corresponding eigenspaces:

Row reducing $A - 0I = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ leads to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Thus the corresponding eigenspace is 1-dimensional with basis $\left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$.

Row reducing $A - 1I = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}$ leads to $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Thus the corresponding eigenspace is 2-dimensional with basis $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Therefore the basis $\beta = \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ makes $[L_A]_\beta$ a diagonal matrix, and

$$D = [L_A]_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Furthermore, $Q = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$.

14. (0 points) Let $A \in M_{n\times n}(\mathbb{R})$ have the property that $A^2 = 3A$. Prove that the only numbers that could possibly be eigenvalues of A are 0 and 3. (Hint: Suppose that λ is an eigenvalue corresponding to eigenvector v and consider A^2v.)

Solution: Suppose that λ is an eigenvalue corresponding to eigenvector v. Then $Av = \lambda v$. Notice that

$$A^2v = A(Av) = A(\lambda v) = \lambda(Av) = \lambda^2 v.$$

However, since $A^2 = 3A$, we have

$$\lambda^2 v = A^2 v = 3Av = 3(\lambda v).$$

Thus $\lambda^2 = 3\lambda$, and therefore $\lambda = 0$ or $\lambda = 3$. Clearly these are the only choices for eigenvalues of A.