1 Example of Proper Proofs

To give you an idea of what is expected when you prove that a set is a vector space, here are two versions of a proof to the same problem. Either one of these would be considered correct, proper proofs. We first include the properties of a vector space for convenience.

Definition 1. *A* ***vector space*** *V over a field F is a set of elements with two defined operations called* ***addition*** *and* ***scalar multiplication*** *such that the following properties hold.*

(C1) *(Closure of Addition) If $x, y \in V$ then $x + y \in V$.*

(C2) *(Closure of Scalar Multiplication) If $x \in V$ and $c \in F$ then $cx \in V$.*

(V1) *(Commutativity of Addition) For all $x, y \in V$, $x + y = y + x$.*

(V2) *(Associativity of Addition) For all $x, y, z \in V$, $(x + y) + z = x + (y + z)$.*

(V3) *(Additive Identity) There is an element called the* ***zero vector*** *and denoted $\vec{0}$ such that $x + \vec{0} = x$ for all $x \in V$.*

(V4) *(Additive Inverse) For each element $x \in V$ there is an element $y \in V$ such that $x + y = \vec{0}$. This y is called the* ***additive inverse*** *of x and is usually denoted $-x$.*

(V5) *(Scalar Identity) For each $x \in V$, $1x = x$.*

(V6) *(Scalar Associativity) For all $x \in V$ and $a, b \in F$, $(ab)x = a(bx)$.*

(V7) *(Scalar Distribution) For all $x, y \in V$ and $a \in F$, $a(x + y) = ax + ay$.*

(V8) *(Vector Distribution) For all $x \in V$ and $a, b \in F$, $(a + b)x = ax + bx$.*

Proposition 2. *The set of real-valued even functions defined defined for all real numbers with the standard operations of addition and scalar multiplication of functions is a vector space.*

Before we begin the proof, let us recall a few things.

1. The **standard operations for functions** are defined as

$$(f + g)(x) = f(x) + g(x) \text{ and } (cf)(x) = c[f(x)].$$

2. A function is called **even** if $f(-x) = f(x)$ for all x.

3. Remember the following distinction: **f is a *function*, but $f(x)$ is a *real number*.** That is, once we plug something into our function, we get a real number not a function. For example, if x is a real number, then x^2 is a real number, not a function. This is subtle but extremely important. For example, we know real numbers commute and thus know that $f(x) + g(x) = g(x) + f(x)$. However, we *must prove* that $f + g = g + f$

Remember all of these as we progress through the proof of the proposition. We will refer to them throughout. The blue text above the equalities is the justification for that equality. For example, when you see "by 3.", we are saying that this step is valid because we are simply dealing with real numbers, not functions.

1.1 Proof of Proposition 2

Proof. **(C1)** Let f, g be even functions. Then $f(-x) = f(x)$ and $g(-x) = g(x)$. ***Is*** $f + g$ ***even***?

$$(f+g)(-x) \overset{\text{by 1.}}{=} f(-x) + g(-x) \overset{f,g\text{ even}}{=} f(x) + g(x) \overset{\text{by 1.}}{=} (f+g)(x)$$

Thus $f + g$ is even.

(C2) Let f be even and $c \in \mathbb{R}$. ***Is*** cf ***even***?

$$(cf)(-x) \overset{\text{by 1.}}{=} c[f(-x)] \overset{f\text{ even}}{=} c[f(x)] \overset{\text{by 1.}}{=} (cf)(x)$$

Thus cf is even.

(V1) Let f, g be even functions. ***Does*** $f + g = g + f$?

$$(f+g)(x) \overset{\text{by 1.}}{=} f(x) + g(x) \overset{\text{by 3.}}{=} g(x) + f(x) \overset{\text{by 1.}}{=} (g+f)(x)$$

Thus $f + g = g + f$.

(V2) Let f, g, h be even functions. ***Does*** $(f + g) + h = f + (g + h)$?

$$\begin{aligned}[(f+g)+h](x) &\overset{\text{by 1.}}{=} (f+g)(x) + h(x) \overset{\text{by 1.}}{=} [f(x) + g(x)] + h(x) \\ &\overset{\text{by 3.}}{=} f(x) + [g(x) + h(x)] \overset{\text{by 1.}}{=} f(x) + (g+h)(x) \overset{\text{by 1.}}{=} [f + (g+h)](x)\end{aligned}$$

Thus $(f + g) + h = f + (g + h)$.

(V3) ***Is there a zero vector?*** We know the constant function $z(x) = 0$ for all x is the additive identity, we just have to verify that it is even (so that it will be in this set of even functions.)

$$z(-x) \overset{\text{by def}}{=} 0 \overset{\text{by def}}{=} z(x)$$

Thus $z(x) = 0$ is even.

(V4) Let f be even. ***Is*** $-f$ ***even?***

$$(-f)(-x) \overset{\text{by 1.}}{=} -[f(-x)] \overset{f\text{ even}}{=} -[f(x)] \overset{\text{by 1.}}{=} (-f)(x)$$

Thus $-f$ is even.

(V5) Let f be even. ***Does*** $1f = f$?

$$(1f)(x) \overset{\text{by 1.}}{=} 1[f(x)] \overset{\text{by 3.}}{=} f(x)$$

Thus $1f = f$.

(V6) Let f be even and $a, b \in \mathbb{R}$. ***Does*** $(ab)f = a(bf)$?

$$[(ab)f](x) \overset{\text{by 1.}}{=} (ab)[f(x)] \overset{\text{by 3.}}{=} a(b[f(x)]) \overset{\text{by 1.}}{=} a[(bf)(x)] \overset{\text{by 1.}}{=} [a(bf)](x)$$

Thus $(ab)f = a(bf)$.

(V7) Let f, g be even and $a \in \mathbb{R}$. ***Does*** $a(f+g) = af + ag$)?

$$[a(f+g)](x) \overset{\text{by 1.}}{=} a[(f+g)(x)] \overset{\text{by 1.}}{=} a[f(x)+g(x)] \overset{\text{by 3.}}{=} a[f(x)]+a[g(x)] \overset{\text{by 1.}}{=} (af)(x)+(ag)(x)] \overset{\text{by 1.}}{=} [af+ag)](x)$$

Thus $a(f+g) = af + ag$.

(V8) Let f be even and $a, b \in \mathbb{R}$. ***Does*** $(a+b)f = af + bf$)?

$$[(a+b)f](x) \overset{\text{by 1.}}{=} (a+b)[f(x)] \overset{\text{by 3.}}{=} a[f(x)] + b[f(x)] \overset{\text{by 1.}}{=} (af)(x) + (bf)(x)] \overset{\text{by 1.}}{=} [af+bf)](x)$$

Thus $(a+b)f = af + bf$.

We have shown that all of the properties of a vector space are true for the set of even functions. Therefore, this set is a vector space. □

1.2 Alternate Proof of Proposition 2

Now we will see a shorter, alternate proof. We make use of the fact that we already know that the set of real-valued functions $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is a vector space. For example, because *ALL* functions satisfy commutativity of addition, we know that even functions do to.

Proof. The set of even functions is a subset of the vector space $\mathcal{F}(\mathbb{R}, \mathbb{R})$. Thus we know that even functions will automatically satisfy (V1), (V2), and (V5)-(V8) because these are true for $\mathcal{F}(\mathbb{R}, \mathbb{R})$. The only properties left to show are (C1), (C2), (V3), and (V4).

(C1) Let f, g be even functions. Then $f(-x) = f(x)$ and $g(-x) = g(x)$. ***Is*** $f + g$ ***even***?

$$(f+g)(-x) \overset{\text{by 1.}}{=} f(-x) + g(-x) \overset{f,g\text{ even}}{=} f(x) + g(x) \overset{\text{by 1.}}{=} (f+g)(x)$$

Thus $f + g$ is even.

(C2) Let f be even and $c \in \mathbb{R}$. ***Is*** cf ***even***?

$$(cf)(-x) \overset{\text{by 1.}}{=} c[f(-x)] \overset{f\text{ even}}{=} c[f(x)] \overset{\text{by 1.}}{=} (cf)(x)$$

Thus cf is even.

(V3) ***Is there a zero vector of*** $\mathcal{F}(\mathbb{R}, \mathbb{R})$ ***even?*** We know the constant function $z(x) = 0$ for all x is the additive identity of $\mathcal{F}(\mathbb{R}, \mathbb{R})$, we just have to verify that it is even.

$$z(-x) \overset{\text{by def}}{=} 0 \overset{\text{by def}}{=} z(x)$$

Thus $z(x) = 0$ is even.

(V4) Let f be even. ***Is*** $-f$ ***even?***

$$(-f)(-x) \overset{\text{by 1.}}{=} -[f(-x)] \overset{f\text{ even}}{=} -[f(x)] \overset{\text{by 1.}}{=} (-f)(x)$$

Thus $-f$ is even.

We have shown that all of the properties of a vector space are true for the set of even functions. Therefore, this set is a vector space. □