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Understand the Principles and Properties of Coordinate Geometry

Exemplar 1:
- Applying the principles of distance, midpoint, slope, parallelism, and perpendicularity to characterize coordinate geometry

- Distance:
  - Distance Formula = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
  - The formula is used to find the distance between two points situated in $A(x_1, y_1)$ and $B(x_2, y_2)$
- Midpoint:
  - $M(x, y) = (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$
  - The formula is used to find the coordinates at which a line is divided into two equal halves
- Slope:
  - $m = \frac{y_2-y_1}{x_2-x_1}$
  - The formula is used to find the slope of a line with points $A(x_1, y_1)$ and $B(x_2, y_2)$
- Parallelism:
  - Two lines are said to be parallel if they have equal slope
  - If two lines are parallel AND have the same y-intercept, then the lines are the same line.
- Perpendicularity:
  - Two lines are said to be perpendicular if the slope of one line is the negative reciprocal of the other
  - The the product of the slopes equal -1 when the lines are perpendicular

Exemplar 2:
- Using coordinate geometry to prove theorems about geometric figures
Using the above principles and the transformations in Exemplar 4, the following theorems about geometric figures can be proven.
Theorems of Triangles:

- Measures of interior angles of a triangle sum to 180°
- Base angles of isosceles triangles are congruent
- The segment joining midpoints of two sides of a triangle is parallel to the third side and half the length
- The medians of a triangle meet at a point

Theorems of Parallelograms:

- A diagonal of a parallelogram divides it into two congruent triangles.

- In a parallelogram, opposite sides are equal
- In a parallelogram, opposite angles are equal
- The diagonals of a parallelogram bisect each other

Theorems of Circles:

- Two equal chords of a circle subtend equal angles at the centre of the circle
- The perpendicular to a chord bisects the chord if drawn from the centre of the circle
- Equal chords of a circle are equidistant (equal distance) from the centre of the circle

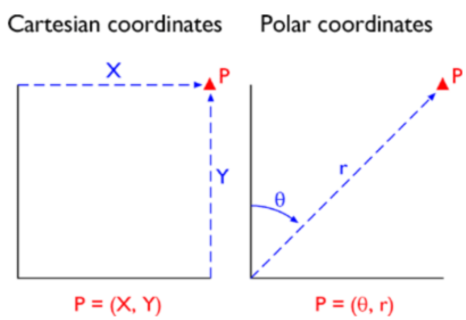
Theorems of Parabolas:

- The distance from any point on the parabola to the focus is equal to the distance from that point to the directrix

Theorems of Hyperbolas:

- If S, S' are the foci and P is any point on a hyperbola, show that SP, S'P are equally inclined to the tangent at P

Exemplar 3:
- Represent two and three dimensional geometric figures in various coordinate systems



<u>2-Dimensional Coordinate Systems:</u>

Polar Grid:

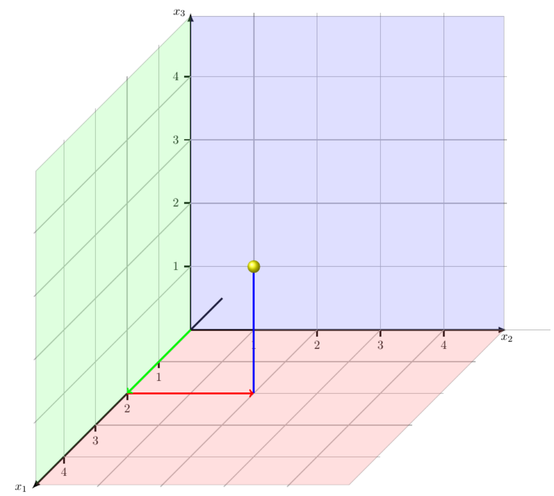
- r-value: distance from origin (radius); directed line segment
- θ: direction of r

Conversions between coordinate systems:

- Polar(r, θ) $\rightarrow$ Cartesian(x, y): Use $\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta$ for x and $\sin\theta = \frac{x}{y} \rightarrow y = r\sin\theta$ for y.
- Cartesian$(x, y) \rightarrow$ Polar(r, θ): Use $\tan\theta = \frac{y}{x}$ to get the direction of r. Use $r^2 = x^2 + y^2$ to get the r-value

3-Dimensional Coordinate Systems:

Coordinate system:



- The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R}\{(x, y, z)|x, y, z \in \mathbb{R}\}$: The set of all ordered triples of real
- In $\mathbb{R}^3$, we deal with three planes and coordinates are (x, y, z).
- The figure to the right represents what a point in $\mathbb{R}^3$ would look like
- The distance between two points, P_1, P_2, in $\mathbb{R}^3$ uses the distance formula defined as, $|P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

Exemplar 4:

- Analyzing and applying transformations in the coordinate plane

Reflections
Over the x-axis: $(x, y) \to (x, -y)$
Over the y-axis: $(x, y) \to (-x, y)$
Over the line $y = x$: $(x, y) \to (y, x)$
Through the origin: $(x, y) \to (-x, -y)$
Translations
$(x, y) \to (x \pm h, y \pm k)$
where h =horizontal shift, y =vertical shift
Rotation
Rotation 90°: $(x, y) \to (-y, x)$
Rotation 180°: $(x, y) \to (-x, -y)$
Rotation 270°: $(x, y) \to (y, -x)$
Dilation
If n is the factor of dilation
$(x, y) \to (nx, ny)$ height

Exemplar 5:

- Applying the distance formula to derive the equation of a conic section

Conic Sections: Circles, Parabolas, Ellipses & Hyperbolas

A conic section results from a right circular cone intersecting with a plane. The formulas for the conic sections are derived by using the distance formula, which was derived from the Pythagorean Theorem. If you know the distance formula and how each of the conic sections is defined, then deriving their formulas becomes simple. Simplifying the algebraic equation; squaring, combining like terms, factoring, and substituting is all it takes to be successful.
Deriving the equations of conic sections using the distance formula:

$$\text{Distance Formula} = \sqrt{(x_1 + x_2)^2 + (y_1 - y_2)^2}$$

Circles: A circle is a set of points in a plane that are equidistant from a fixed point, called the center.

- An equation of a circle can be derived from the distance formula by squaring both sides to obtain $x^2 + y^2 = r^2$
- Let $C(h, k)$ be the center of the circle, then $(x - h)^2 + (y - k)^2 = r^2$

Parabolas: A parabola is a set of points, P, whose distance from a fixed point, called the focus, is equal to the perpendicular distance from P to a line, called the directrix.

- Using the distance formula, we can find an equation for the distance from the focus to P and another equation from P to the directrix
- By definition of a parabola, we set the distances equal
- After squaring both sides, expanding, and subtracting terms on both sides, we obtain $y = \frac{1}{4c}x^2$; where c is the distance from the focus to the vertex
- Let $V(h, k)$ be the vertex, then $y - k = \frac{1}{4c}(x - h)^2$
- The equation of the line representing the directrix is $y = k - c$

Ellipses: An ellipses is a set of points, P, in a plane for each of which the sum of the distances from two fixed points, called the foci, is a constant, $2a$

- The distances from these two foci, $F1(-c, 0)$ and $F2(c, 0)$, to a point P on the curve are called the focal radii of P.
- The transverse axis has length $2a$ and the conjugate axis has length $2b$
- The sum of the focal radii is $2a$. By definition, $d(p, F1) + d(P, F2) = 2a$
- After a series of algebraic simplifying, we obtain $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
- Let $C(h, k)$ be the center, then $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

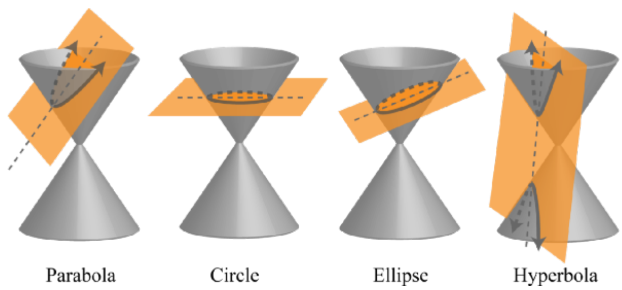
Hyperbolas: A Hyperbola is the set of points in a plane such that for each point, the absolute value of their difference of its distances, called the focal radii, from two fixed points, called the foci, is a constant, $2a$

- By defintion, $d(P, F) - d(P, F') = 2a$
- After a series of algebraic simplifying, we obtain $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
- Let the origin be (h, k), then $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

Exemplar 6:

- Modeling and solving problems using conic sections

The standard form of an equation of a conic section is $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where A, B, C, D, E, F are real numbers and $A \neq 0, B \neq 0, C \neq 0$.



Consider $B^2 - 4AC$,

- If $A = C$ and $B = 0$, then the conic section is a circle.
- If $B^2 - 4AC = 0$, then the conic section is a parabola
- If $B^2 - 4AC < 0$, then the conic section is an ellipse
- If $B^2 - 4AC > 0$, then the conic section is a hyperbola.

Eccentricity: The ratio of the distance of point P from the focus to its distance from the directrix is a constant e known as the eccentricity. In other words, eccentricity is a measure of the deviation of the ellipse from being circular.

- If $e = 0$, then the conic is a circle
- If $0 < e < 1$, then the conic is an ellipse
- If $e = 1$, then the conic is a parabola
- And if $e > 1$, then the conic is a hyperbola

Using the formulas for each conic section in vertex/center/origin form in Exemplar 5, we can graph each conic section.

Resources

https://byjus.com/maths/conic-sections/
http://www.hanlonmath.com/pdfFiles/640Conicsections.PDF
https://brilliant.org/wiki/convert-cartesian-coordinates-to-polar/
Sections in textbook, 10.2, 10.3
Picture retrieved from Bing Images

Questions:

1)Given endpoints of segment $\overline{AB}$, $A(4,5)$ and $B(6,7)$, find

a) The midpoint

b) The distance

c) The slope

d) Using the slope in part c), give a slope of a line parallel and a slope of a line perpendicular to $\overline{AB}$

Solution:
a): Given $(x_1, y_1) = (4,5)$, $(x_2, y_2) = (6,7)$

According to the formula we can find the midpoint (x, y):

$$\begin{aligned} M(x,y) &= (\frac{(x_1+x_2)}{2}, \frac{(y_1+y_2)}{2}) \\ &= (\frac{(4+6)}{2}, \frac{(5+7)}{2}) \\ &= (5,6) \end{aligned}$$

b): Given $(x_1, y_1) = (4,5)$, $(x_2, y_2) = (6,7)$

According to the formula we can find the distance (x, y):

$$\begin{aligned} d(x,y) &= \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} \\ &= \sqrt{(6-4)^2 + (7-5)^2} \\ &= \sqrt{8} = 2\sqrt{2} \end{aligned}$$

c): Given $(x_1, y_1) = (4,5)$, $(x_2, y_2) = (6,7)$

According to the formula we can find the slope (x, y):

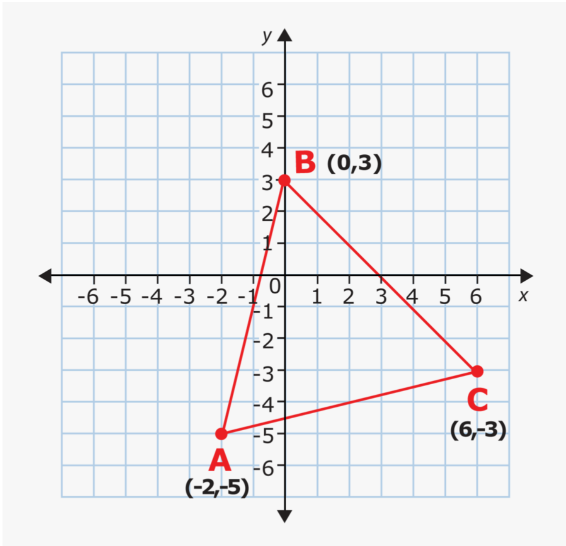
$$\begin{aligned} m(x,y) &= \frac{(y_2-y_1)}{(x_2-x_1)} \\ &= \frac{(7-5)}{(6-4)} \\ &= \frac{2}{2} = 1 \end{aligned}$$

d) Using $m = 1$ from part c):

Parallel lines are two lines that have the same slope, thus, a slope of a line parallel to $\overline{AB}$ is $m = 1$.

Perpendicular lines are two lines that have a product of the slope equal to -1, thus, a slope of a line perpendicular to $\overline{AB}$ is $m = -1$.

2) Given $\triangle ABC$, prove that the segment joining midpoints of $\overline{AB}$ and $\overline{BC}$ of $\triangle ABC$ is parallel to the segment $\overline{AC}$.



Solution:

Proof. Given $\triangle ABC$ with points $A(-2,-5)$, $B(0,3)$, and $C(6,-3)$. Using the midpoint formula we can find the midpoint of $\overline{AB}$ and $\overline{BC}$. Let $A(-2,-5) = (x_1, y_1)$ and $B(0,3) = (x_2, y_2)$, then

$$\begin{aligned} M(x,y) &= (\frac{(x_1 + x_2)}{2}, \frac{(y_1 + y_2)}{2}) \\ &= (\frac{(-2+0)}{2}, \frac{(-5+3)}{2}) \\ &= (-1,-1). \end{aligned}$$

The midpoint of $\overline{AB} = (-1,-1)$, call it D.
Now, let $B(0,3) = (x_1, y_1)$ and $C(6,-3) = (x_2, y_2)$, then

$$\begin{aligned} M(x,y) &= (\frac{(x_1 + x_2)}{2}, \frac{(y_1 + y_2)}{2}) \\ &= (\frac{(0+6)}{2}, \frac{(3+-3)}{2}) \\ &= (3,0). \end{aligned}$$

The midpoint of $\overline{BC} = (3,0)$, call it F.
Consider $\overline{DF}$. Using the slope formula, we can find the slope of $\overline{DF}$. Let $D(-1,-1) = (x_1, y_1)$ and $F(3,0) = (x_2, y_2)$, then

$$\begin{aligned} m =& \frac{y_2 - y_1}{x_2 - x_1} \\ =& \frac{0 - (-1)}{3 - (-1)} \\ =& \frac{1}{4}. \end{aligned}$$

The slope of $\overline{DF} = \frac{1}{4}$.
Now use the slope formula to find the slope of $\overline{AC}$. Let $A(-2,-5) = (x_1, y_1)$ and $C(6,-3) = (x_2, y_2)$, then

$$\begin{aligned} m =& \frac{y_2 - y_1}{x_2 - x_1} \\ =& \frac{(-3) - (-5)}{6 - (-2)} \\ =& \frac{2}{8} = \frac{1}{4}. \end{aligned}$$

The slope of $\overline{AC} = \frac{1}{4}$.
Notice that $m(\overline{DF}) = m(\overline{AC}$. This implies that $\overline{DF} \parallel \overline{AC}$. Hence, the segment joining midpoints of $\overline{AB}$ and $\overline{BC}$ of $\triangle ABC$ is parallel to the segment $\overline{AC}$ $\square$

3) Convert the locus $(x(t), y(t))$ in Cartesian coordinates to the locus in polar coordinates $(r(t), \theta(t))$ where $x = \sqrt{t}\cos t$ and $y = \sqrt{t}\sin t$.

Solution:
By the Pythagorean theorem, we have

$$\begin{aligned} r(t) =& \sqrt{x^2 + y^2} \\ =& \sqrt{\left(\sqrt{t}\cos t\right)^2 + \left(\sqrt{t}\sin t\right)^2} \\ =& \sqrt{t}\left((\cos t)^2 + (\sin t)^2\right) \\ =& \sqrt{t}. \end{aligned}$$

By the tangent function, we can calculate $\theta(t)$ for the case $\cos t > 0$:

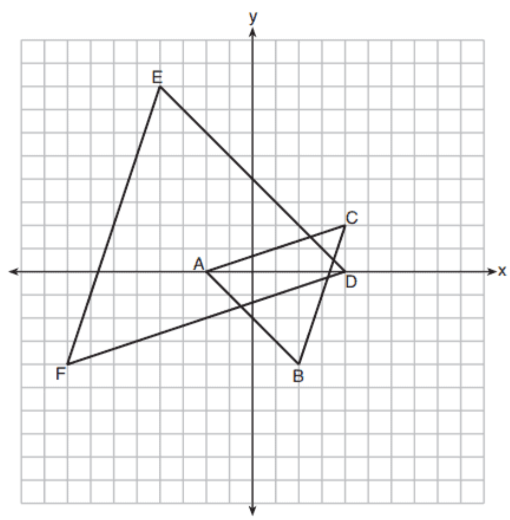
$$\begin{aligned} \theta =& \arctan \frac{\sqrt{t}\sin t}{\sqrt{t}\cos t} \\ =& \arctan \frac{\sin t}{\cos t} \\ =& \arctan \tan t \\ =& t \end{aligned}$$

and for the case $\cos t < 0$:

$$\begin{aligned}\theta =& \arctan \frac{\sqrt{t}\sin t}{\sqrt{t}\cos t} + \pi \\ =& \arctan \frac{\sin t}{\cos t} + \pi \\ =& \arctan \tan t + \pi \\ =& (t - \pi) + \pi \\ =& t.\end{aligned}$$

Thus, our answer is $(r(t), \theta(t)) = (\sqrt{t}, t)$.

4) On the set of axes below, $\triangle ABC$ has vertices at $A(-2, 0)$, $B(2, -4)$, $C(4, 2)$, and $\triangle DEF$ has vertices at $D(4, 0)$, $E(-4, 8)$, $F(-8, -4)$.



Which sequence of transformations will map $\triangle ABC$ onto $\triangle DEF$?

(1) a dilation of $\triangle ABC$ by a scale factor of 2 centered at point A
(2) a dilation of $\triangle ABC$ by a scale factor of $\frac{1}{2}$ centered at point A
(3) a dilation of $\triangle ABC$ by a scale factor of 2 centered at the origin, followed by a rotation of 180° about the origin
(4) a dilation of $\triangle ABC$ by a scale factor of $\frac{1}{2}$ centered at the origin, followed by a rotation of 180° about the origin

Solution:
(4) A dilation of $\triangle ABC$ by a scale factor of $\frac{1}{2}$ centered at the origin, followed by a rotation of 180° about the origin.

5) Derive the equation of a parabola using the distance formula

Solution:
Using the distance formula, we can find an equation for the distance from the focus to P and another equation from P to the directrix. Let the focus be $F(0, c)$, a point on the parabola be $P(x, y)$, and directrix be $D(x, -c)$. Using the distance formula, we obtain

$$FP = \sqrt{(x-0)^2 + (y-c)^2}$$
$$PD = \sqrt{(x-x)^2 + (y+c)^2}.$$

By definition of a parabola, we know $FP = PD$. Then,

$$\sqrt{(x-0)^2 + (y-c)^2} = \sqrt{(x-x)^2 + (y+c)^2}$$
$$x^2 + (y-c)^2 = 0^2 + (y+c)^2$$
$$x^2 + y^2 - 2yc + c^2 = y^2 + 2yc + c^2$$
$$x^2 - 2yc = 2yc$$
$$x^2 = 4yx$$
$$y = \frac{1}{4c}x^2.$$

This is an equation of a parabola derived from the distance formula.

6) Given an ellipse whose foci are at $(\pm 4, 0)$ and the eccentricity is $\frac{1}{3}$. Find the equation of the ellipse.

Solution:
By the coordinates of focus, we get that the ellipse is a horizontal ellipse whose major axis lies on the x-axis.
Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } a^2 > b^2$$

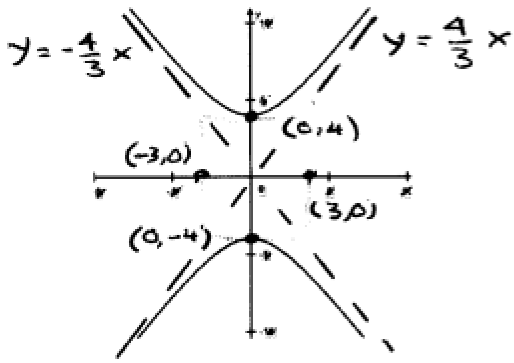
For an ellipse, the eccentricity $e = \frac{c}{a}$. This suggests, $a = \frac{c}{e}$ where $(\pm c, 0)$ is the focus.
Therefore, $a = \dfrac{4}{\frac{1}{3}} = 12$.

Using, $c^2 = (a^2–b^2)$
$b^2 = (a^2–c^2) = 122–42 = 128$

Hence, the equation of the ellipse is $\dfrac{x^2}{144} + \dfrac{y^2}{128} = 1$.

7) Sketch the graph $\frac{y^2}{4^2} - \frac{x^2}{3^2} = 1$

Solution:
Given $\frac{y^2}{4^2} - \frac{x^2}{3^2} = 1$. Therfore, $a = 4$ and $b = 3$. By inspection, the y-intercepts are 4 and -4. There are no x-intercepts. The asympotopes are graphs $y = \frac{4x}{3}$ and $y = -\frac{4x}{3}$. Thus,



8) What do you thing we use conic sections for? That is, are there real-life applications of conic sections?

Solution:
Sample answers

- The paths of the planets around the sun are ellipses with the sun at one focus
- Parabolic mirrors are used to converge light beams at the focus of the parabola
- Parabolic microphones perform a similar function with sound waves