213 Problem Set 8 Solutions

§3.9 3. We start with $x^2y = y - 7$ and differentiate both sides. The left side is the product rule, and the right is quite short $2xy + x^2\frac{dy}{dx} = \frac{dy}{dx}$. We then move the x^2 term to the right to get $2xy = \frac{dy}{dx} - x^2\frac{dy}{dx}$, and factor from the right $2xy = \frac{dy}{dx}(1 - x^2)$ and finally divide, $\frac{dy}{dx} = \frac{2xy}{1-x^2}$.

§3.9 25. Hey, it's something about money! (I did actually select it mostly for that reason). I'd prefer l for labour and c for capital. It's worth being clear, the number of cars is 360. They are not changing how many cars to make, it is constant. So, we start with the equation $30l^{1/3}c^{2/3} = 360$. Notice in these models, the managers either spend money on labour or capital, and the money on capital has more of an effect. You know better than I do if this is reasonable. Because of this, as l changes, c changes. We are looking for $\frac{dc}{dl}$, the rate of change of c with respect to l. We differential implicitly (using the product and chain rule on the left, remembering it really starts with $30(l^{1/3})(c(l)^{2/3}) = 360$): $10l^{-2/3}(c^{2/3}) + 20l^{1/3}c^{-1/3}\frac{dc}{dl} = 0$. Solving we get $\frac{dc}{dl} = -\frac{10l^{-2/3}(c^{2/3})}{20l^{1/3}c^{-1/3}} = -\frac{c}{2l}$. That's the general result. In in the particular case of \$27,000 on labour and \$8,000 on capital this computes to $-\frac{8}{54} = -\frac{4}{27} \simeq -0.148$, which tells us that when the vales are close to \$27,000 on labour and \$8,000 on capital, if an extra dollar is spent on labour, it only saves about 15 cents on capital.

§4.2 8. Aside from the fact that you and your friend need to talk before you set out, because this plan seems doomed, I'm going to call the east distance e and the north distance n. (Hm, I could have used y for you and f your friend.) I will call the distance between d. We've done examples like this. We are told $e = 4$ mi., $\frac{de}{dt} = 16$ mph, and $\frac{dn}{dt} = 12$ mph. We have a right triangle with $e(t)^2 + n(t)^2 = d(t)^2$. We differentiate implicitly to get $2e\frac{de}{dt} + 2n\frac{dn}{dt} = 2d\frac{dd}{dt}$. We need some more information. Surprisingly, we need the time. You are traveling 16 mph, and you traveled 4 miles, so it must be that $t = \frac{1}{4}$ hour. In that time your friend has traveled $n = \frac{12}{4} = 3$ miles northbound. We can use this information to find d from $e(t)^2 + n(t)^2 = d(t)^2$ to get $d = 5$ miles. So, now we put all this into: $2e\frac{de}{dt} + 2n\frac{dn}{dt} = 2d\frac{dd}{dt} = 2(4)(16) + 2(3)(12) = 2(5)\frac{dd}{dt}$. Lots of numbers and one thing we don't know. Looks good. We have $128 + 72 = 10\frac{dd}{dt}$. And like magic the numbers work out great, $\frac{dd}{dt} = \frac{200}{10} = 20$ mph for the speed that you and your misguided friend are moving apart.

§4.2 10. Kindly there's a diagram of the situation. I hope you see that it involves similar triangles. Don't feel bad if you do not. (But, I hope you asked me or someone about it if so.). The whole picture is a triangle, and the small triangle with height the person and base indicated by x (why x?), which is the length of the shadow (so really ought to be s, which is what I will use so that I remember what it means). These two triangles are similar, so their sides are proportional. The picture is, in fact, rather misleading because what is labeled "10" is changing, but the post height isn't. There are two variables here, one is s, and the other is the distance from the person to the pole. I will call this p, for their person or pole or both. The question is extra tricky (sorry) because the question asks for the rate that the tip of the shadow moves away from the pole. We'll need yet one more variable, $w = p + s$, the whole distance (that's why I used w) from the end of shadow to the pole. The goal is $\frac{dw}{dt}$. We're told $p = 10$ feet, and $\frac{dp}{dt} = 3$ ft/sec. We also know the post is ten feet and the person is six. The similar triangles give us part-to-whole proportions: $\frac{\text{post height}}{\text{whole distance post to shadow}} = \frac{\text{person height}}{\text{shadow length}}$, so $\frac{10}{w} = \frac{6}{s}$. To make this simpler for the calculus, we will cross multiply to get $10s = 6w$. We now differentiate both to get $\frac{dw}{dt} = \frac{dp}{dt} + \frac{ds}{dt}$ and $10\frac{ds}{dt} = 6\frac{dw}{dt}$. We know $\frac{dp}{dt} = 3$ ft/sec and we use it to get $\frac{dw}{dt} = 3 + \frac{ds}{dt}$. Substitute that into the second one to get $10\frac{ds}{dt} = 6\left(3 + \frac{ds}{dt}\right) = 18 + 6\frac{ds}{dt}$. Hence $\frac{ds}{dt} = 4.5$ mph and $\frac{dw}{dt} = 3 + \frac{ds}{dt} = 7.5$ mph as the speed that the tip of the shadow is moving from the base of the pole.