222 §8.7

In this section we're dealing with borderline integrals: integrals that have problems because of the boundaries. Here's the most obvious example: $\int_1^\infty \frac{1}{x^2}dx$. The problem, obviously, is with the upper bound. How to deal with it? (1) How do we always deal with infinity? Yes, take a limit. And, yes, that's about the most interesting question I can think of. I would write explanations, but they would be silly. (2) Follow the steps here:

$$\int_1^\infty \frac{1}{x^2}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}dx = \lim_{t\to\infty}\left[-\frac{1}{x}\right|_1^t = \lim_{t\to\infty} -\frac{1}{t} - (-1) = 0+1 = 1$$

That result may surprise you - in all that infinite length, only an area of 1. In fact, that's the whole idea for this section. Nice to end on something simple, eh? Here's something interesting, compare the graph of $\frac{1}{x}$ to $\frac{1}{x^2}$. (3) Which has more area under the graph from 1 to ∞? How much more? (4) Follow the steps above to compute $\int_1^\infty \frac{1}{x}dx$. Ok, now, how much more? Interesting.

What about the other side? $\int_0^1 \frac{1}{x}dx$. Now the problem is with the lower bound. (5) Do as much of this as you can without reading what comes next, then check with this:

$$\int_0^1 \frac{1}{x}dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x}dx = \lim_{t\to 0^+}\left[\ln x\right|_t^1 = \lim_{t\to 0^+} 0 - \ln t = \infty$$

Here's another one, what about $\int_{-\infty}^\infty \frac{1}{1+x^2}dx$? This has boundary problems on both sides. The best way to deal with this dual-challenge is to separate it into two separate integrals, and just deal with them entirely independently of each other, get two separate answers, and then, if they are both finite, then add them in the end. In this case, there are lots of choices between the two limits, $x=0$ is probably the most natural (although curiously every point is half-way in between). (6) Show all work for $\int_{-\infty}^0 \frac{1}{1+x^2}dx + \int_0^\infty \frac{1}{1+x^2}dx$. You should get $\frac{\pi}{2}$. (7) Show enough work to explain why this approach does *not* produce $\int_{-\infty}^\infty x dx = 0$.

Maybe this one will be more interesting: $\int_1^4 \frac{1}{(x-3)^2}dx$. Here's something different: (8) Make a graph of $\frac{1}{(x-3)^2}$ from 1 to 4. (9) How do you feel about this answer:

$$\int_1^4 \frac{1}{(x-3)^2}dx = \left[-\frac{1}{x-3}\right|_1^4 = -1-(\frac{1}{2}) = -\frac{3}{2}?$$

Presuming you don't like it, why not? In other words, why is the answer unacceptable? Next, what is wrong in the work? And, how do you fix it? Make sense out of this situation.

In the end there's not so much here. You probably appreciate that at this point, and I do too. Work for an upcoming exam.