223 Assignment 3 Solutions

§10.1 #12 a. The domain of $g(x,y) = \frac{xy}{x^2+y^2}$ is all (x,y), except $(0,0)$.

b. $x = 0$: $\lim\limits_{y\to 0} = \frac{0y}{0+y^2} = 0$.

$y = x$: $\lim\limits_{x\to 0} = \frac{xx}{x^2+x^2} = \frac{1}{2}$.

$y = 2x$: $\lim\limits_{x\to 0} = \frac{x(2x)}{x^2+(2x)^2} = \frac{2}{5}$.

c. I can now say that $\lim\limits_{(x,y)\to(0,0)} g(x,y)$ does not exist.

d. g is very far from being continuous at $(0,0)$. Remember for this to be the case we need: i. g defined at $(0,0)$ (it's not (a)), ii. the limit defined at $(0,0)$ (It's not (c)), and iii. them to be equal (this is meaningless at this point). And, if you're hoping, we don't get that two things that don't exist are therefore equal.

e. Make maple contour plots, and surface graphs. And notice that there's some serious craziness going on at the origin.

#14. a. I'm trying to keep it simple, and we just did the previous question, so I'll let $p(x,y) = \begin{cases} g(x,y) = \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ \pi^2 & \text{if } (x,y) = (0,0) \end{cases}$. Adding in the extra point does not change the existence of the limit from the last question.

b. This is the important and surprising effect of the example in Activity 10.2. Quoting that example would be fine, but I haven't written up the work there, so I will create a similar but slightly new example with the same ideas. Let $q(x,y) = \frac{x^{5/3}y^2}{x^5+y^3}$. i. What is $\lim\limits_{(x,y)\to(0,0)} q(x,y)$ only considered along $y = mx$? I need notation like in calc I for not just limit from the right, or from the left, but along this path ... hm. Anyway $\lim\limits_{x\to 0} q(x,mx) = \lim\limits_{x\to 0} \frac{x^{5/3}m^2x^2}{x^5+x^3m^3} = \lim\limits_{x\to 0} \frac{x^{2/3}m^2}{x^2+m^3} = (\frac{0}{m^3}) = 0$ (unless $m = 0$, in which case the numerator is constantly zero, so we also get a 0 limit). ii. Now, let $y = x^{5/3}$, hence $y^3 = x^5$. We now take the limit along this path. $\lim\limits_{x\to 0} q(x, x^{5/3}) = \lim\limits_{x\to 0} \frac{x^{5/3}x^{10/3}}{x^5+x^5} = \lim\limits_{x\to 0} \frac{x^5}{x^5+m^5} = \frac{1}{2}$

c. This is impossible. The definition of continuity requires that the limit exists.

d. This is impossible. The limit cannot exist if there are two different values from different paths.

e. (don't forget to turn the page). I'll use a calc I idea here: Let $t(x,y) = \frac{1-x^2y^2}{1-xy}$ That was my first try, but notice that it's not defined for any where on $y = \frac{1}{x}$, which makes the limit undefined along that path. Curses. Well, here's a modification that's just a bit messier: Let $t(x,y) = \frac{1-(1+(x-1)^2+(y-1)^2)^2}{1-(1+(x-1)^2+(y-1)^2)}$. Notice the denominator equals zero at $(1,1)$, so the function is not defined there, but if that's not the case, we can factor the numerator as a difference of squares and cancel the denominator. We are then left with a limit of 2. I am interested to see what answers students produce to this question. There must be something simpler. Upon reflection - I think the first answer is fine. It is similar to $f(x) = \sqrt{x}$ in calc I. At least the answer is the same for that limit as $x \to 0$.

§10.2 #1 The partial differential equation $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = ku$ is used in population modeling. Here $u = u(x,t)$ is the number of individuals of age x at time t, and k is the mortality rate. Show if $a + b = k$, then the function $u(x,t) = e^{ax+bt}$ is a solution to this equation. $\frac{\partial u}{\partial t} = be^{ax+bt}$ and $\frac{\partial u}{\partial x} = ae^{ax+bt}$, so $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = (b+a)e^{ax+bt} = ke^{ax+bt} = ku$, as desired.

#16 $f(x,y) = 8 - x^2 - 3y^2$.

(a) $\frac{\partial f}{\partial x}(x,y) = -2x$, $\frac{\partial f}{\partial y}(x,y) = -6y$

(b) $f(x,1) = 5 - x^2$, $f'(x,1) = -2x$, at $x = 2$ this gives -4, so the tangent line is
$T_{y=1}(t) = (t, 1, -4(t-2)+1)$

(c) $f(2,y) = 4 - 3y^2$, $f'(2,y) = -6y$, at $y = 1$ this gives -6, so the tangent line is
$T_{x=2}(t) = (2, t, -6(t-1)+1)$

(d) $\langle 1, 0, -4\rangle$ and $\langle 0, 1, -6\rangle$, respectively.

(e) $f(2,1) = 1$. $\langle 1,0,-4\rangle \times \langle 0,1,-6\rangle = \langle 4,6,1\rangle$ The plane is then $\langle 4,6,1\rangle \cdot (\mathbf{x} - \langle 2,1,1\rangle) = 0$ i.e. $4x + 6y + z = 15$. Please note for later use that the normal vector here is $\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\rangle$.

(f) The plane found is the tangent plane at the point $(2,1)$.

§10.3 #1 Define $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{(x^2+y^2)} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. Calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ (note: you will need to use the limit definition). Now calculate $\frac{\partial f^2}{\partial x \partial y}(0,0)$, and $\frac{\partial f^2}{\partial y \partial x}(0,0)$. Show they are not equal.

Are mixed partials always equal? Well, we set out … $\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{\frac{h0(h^2-0^2)}{h^2+0^2}}{h} = 0$. Similarly $\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{\frac{0h(0^2-h^2)}{0^2+h^2}}{h} = 0$. Away from there we have $\frac{\partial f}{\partial x} = \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}$ and $\frac{\partial f}{\partial y} = \frac{-xy^4-4x^3y^2+x^5}{(x^2+y^2)^2}$ So we now head for $\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{h\to 0} \frac{\frac{-h^5}{(h^2)^2}-0}{h} = -1$. On the other hand $\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{h\to 0} \frac{\frac{h^5}{(h^2)^2}-0}{h} = 1$. This is what can happen when a function is *not* twice continuously differentiable.

#12 a. $\frac{\partial^2 I}{\partial T^2}(94,75) := \lim_{h\to 0} \frac{\frac{\partial I}{\partial T}(94+h,75) - \frac{\partial I}{\partial T}(94,75)}{h}$. To do this, we first need to estimate $\frac{\partial I}{\partial T}$. So, to that end, $\frac{\partial I}{\partial T}(94,75) := \lim_{h\to 0} \frac{I(94+h,75)-I(94,75)}{h}$. Using h values of -2 and 2, respectively produces 3.5 and 4 apparent degrees F / degree F (note that this is the smallest we can make h from the data). We average these two approximations to get that $\frac{\partial I}{\partial T}(94,75) \simeq 3.75$ apparent degrees F / degree F. We will need other nearby values, so we find $\frac{\partial I}{\partial T}(92.75) \simeq 3.25$, and $\frac{\partial I}{\partial T}(96,75) \simeq 4$ using the same techniques (can only estimate on one side for the second one due to limits of the data). Now we apply these techniques to the first definition to find $\frac{\partial^2 I}{\partial T^2}(94,75) \simeq 0.1875$ apparent degrees F / degree F / degree F. This means when the temperature is near 94° F and the humidity is near 75%, as the temperature goes up one degree F, the rate of increase of the heat index goes up approximately 0.1875 apparent degrees F / degree F. What does that mean? The heat index goes up faster when the temperature is higher, and is going up at this rate.

b. $\frac{\partial^2 I}{\partial H^2}(94,75) := \lim_{h\to 0} \frac{\frac{\partial I}{\partial H}(94,75+h) - \frac{\partial I}{\partial T}(94,75)}{h}$. To do this, we first need to estimate $\frac{\partial I}{\partial H}$. So, to that end, $\frac{\partial I}{\partial H}(94,75) := \lim_{h\to 0} \frac{I(94,75+h)-I(94,75)}{h}$. Using h values of -5 and 5, respectively produces $\frac{\partial I}{\partial T}(94,75) \simeq 0.9$ apparent degrees F / %. We will need other nearby values, so we find $\frac{\partial I}{\partial H}(94.70) \simeq 0.8$, and $\frac{\partial I}{\partial T}(96,80) \simeq 1$ using the same techniques (can only estimate on one side for the first one due to limits of the data). Now we apply these techniques to the first definition to find $\frac{\partial^2 I}{\partial H^2}(94,75) \simeq 0.02$ apparent degrees F / % / %. This means when the temperature is near 94° F and the humidity is near 75%, as the humidity goes up one percent, the rate of increase of the heat index goes up approximately 0.02 apparent degrees F / %. What does that mean? The heat index goes up faster when it is more humid, and is going up at this rate.

c. $\frac{\partial^2 I}{\partial T \partial H}(94,75) := \lim_{h\to 0} \frac{\frac{\partial I}{\partial H}(94+h,75) - \frac{\partial I}{\partial T}(94,75)}{h}$. Thankfully we estimated $\frac{\partial I}{\partial H}(94,75) \simeq 0.9$ apparent degrees F / % in b, but now we need estimates for $\frac{\partial I}{\partial H}(92,75)$ and $\frac{\partial I}{\partial H}(96,75)$; we find 0.7 and 1 apparent degrees F / %, respectively. So, now use the first definition to find $\frac{\partial^2 I}{\partial T \partial H}(94,75) \simeq 0.075$ apparent degrees F / % / degree F. This means when the temperature is near 94° F and the humidity is near 75%, as the temperature goes up one degree, the rate of increase of the heat index goes up approximately 0.075 apparent degrees F / %. What does that mean? The heat index is more sensitive to temperature when it is more humid, and the sensitivity to temperature is going up at this rate.

§10.4 #1 Find a point on the surface $x^2 + y^2 + 3z^2 = 8$ where the tangent plane is parallel to the plane $2x + y + 3z = 0$.

Well, the normal vector to the given plane is $\langle 2,1,3\rangle$. Thus ends the easy part. The surface is not given to us in $f(x,y)$ form, so we must begin by solving for $z = \pm\sqrt{(8-x^2-y^2)/3}$. Is there a better way? I hope so. Remember in calc I how you found $\frac{dy}{dx}$ from $x^2 + y^2 = 1$ using implicit derivatives? Let's try using implicit partial derivatives. First differentiate with respect to x: $2x + 6z\frac{\partial z}{\partial x} = 0$, so $\frac{\partial z}{\partial x} = -\frac{x}{3z}$. Next differentiate with respect to y: $2y + 6z\frac{\partial z}{\partial x}$, and similarly $\frac{\partial z}{\partial y} = -\frac{y}{3z}$. Looking back at §10.2 #4, we see the normal vector for the surface is $\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\rangle$. So, for this surface at any $\langle x,y,z\rangle$ on the surface, a normal vector is given by $\langle \frac{x}{3z}, \frac{y}{3z}, 1\rangle$. Since we are only seeking parallels we scale by $3z$ to get $\langle x, y, 3z\rangle$. We compare this to the normal of the plane $\langle 2, 1, 3\rangle$. Now a lucky turn of events: this seems to imply that $x = 2, y = 1$, and $z = 1$. Happily this is a point on the surface (if it weren't we would need to try $x = 2r, y = r, z = r$ and scale with the equation). Notice one thing more, if they are all negative, then we get the same results. So, we have two points on the surface $\langle 2, 1, 1\rangle$ and $\langle -2, -1, -1\rangle$. Either is fine. You might think for a moment how nasty this all would be if you took partial derivatives directly. Don't think about this too long - it's a mess.

#14 I believe differentials in calc I are merely tangent line approximations and in in calc III they are merely tangent plane approximations. So, I will use the tangent plane and compare to the actual value.

a. $f(x, y) = \cos(x)\sin(2y)$ at $(\frac{\pi}{4}, \frac{\pi}{3})$ is $\frac{\sqrt{6}}{4}$. We want some partial derivatives: $\frac{\partial f}{\partial x} = -\sin(x)\sin(2y)$, $\frac{\partial f}{\partial y} = 2\cos(x)\cos(2y)$. At this point they are $\frac{\partial f}{\partial x}(\frac{\pi}{4}, \frac{\pi}{3}) = -\frac{\sqrt{6}}{4}$ and $\frac{\partial f}{\partial y}(\frac{\pi}{4}, \frac{\pi}{3}) = -\frac{1}{\sqrt{2}}$. The approximation at $(\frac{\pi}{4} - 0.01, \frac{\pi}{3} + 0.1)$ is $\frac{\sqrt{6}}{4} + 0.01\frac{\sqrt{6}}{4} - 0.1\frac{1}{\sqrt{2}} \simeq 0.547785471934$. The actual value is 0.535198212949. Ok, not wildly impressive, not very wrong.

b. $R = \frac{1}{\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}}$. $R(25, 40, 50) = \frac{200}{17}$. $\frac{\partial R}{\partial R_1} = \frac{1}{R_1^2}(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3})^{-2}$, and similarly for the others. So, $\frac{\partial R}{\partial R_1}(25, 40, 50) = \frac{1}{25^2}(\frac{200}{17})^2 = \frac{64}{289}$, and similarly $\frac{\partial R}{\partial R_2}(25, 40, 50) = \frac{1}{40^2}(\frac{200}{17})^2 = \frac{25}{289}$, and $\frac{\partial R}{\partial R_3}(25, 40, 50) = \frac{1}{50^2}(\frac{200}{17})^2 = \frac{16}{289}$ The tangent (hyper-)plane approximation is $\frac{200}{17} + \frac{25}{200}\frac{64}{289} + \frac{40}{200}\frac{25}{289} + \frac{50}{200}\frac{16}{289}$, where the variation from the default is $\frac{25}{200}\frac{64}{289} + \frac{40}{200}\frac{25}{289} + \frac{50}{200}\frac{16}{289} \simeq 0.0588235294118$. For comparison $R(25.125, 40.2, 50.25) \simeq 11.8235294118$, which has a variation of about 0.0588235294118 from $\frac{200}{17}$. Now unlike above this *is* stunning. If someone out there figures out either what I did wrong or why this is so amazing before I do, I will give them (the first to email me) 4 extra points on this assignment. It's good to read solutions. Is it as amazing at the far low endpoint? I'll leave that for you ... so I don't figure this out and steal your points myself.