223 Assignment 5 Solutions

§11.1 #1 Let $I = \int\int_R (x^2+y)dA$ where $R=[0,1]\times[0,2]$

a. Explain why $I \geq 0$: On R, $(x^2+y) \geq 0^2+0=0$, so the integral as well will not be negative.

b. Explain why $I \leq 6$: On R, $(x^2+y) \leq 1^2+2=3$, so the volume is inside of a box of dimensions $1\times 2\times 3$, i.e. of volume 6. Since it is inside the box, it has no more volume than the box.

c. Estimate I by calculating a double midpoint sum with four subdivisions (two in each direction): The midpoints are at $(\frac{1}{4},\frac{1}{2}),(\frac{3}{4},\frac{1}{2}),(\frac{1}{4},\frac{3}{2}),(\frac{3}{4},\frac{3}{2})$. The area of each subdivision is $\frac{1}{2}$. So, the requested volume estimate is $\frac{1}{2}(\frac{9}{16}+\frac{17}{16}+\frac{25}{16}+\frac{33}{16})=\frac{21}{8}$.

§11.1 #13 a. The area of each subdivision is 1, so that will make it easy. We just need to find the sum of the values at the midpoints. The twelve midpoints are $(\frac{1}{2},\frac{1}{2}),(1\frac{1}{2},\frac{1}{2}),(2\frac{1}{2},\frac{1}{2}),(3\frac{1}{2},\frac{1}{2}),(\frac{1}{2},1\frac{1}{2}),(1\frac{1}{2},1\frac{1}{2}),(2\frac{1}{2},1\frac{1}{2}),(3\frac{1}{2},1\frac{1}{2}),(\frac{1}{2},2\frac{1}{2}),(1\frac{1}{2},2\frac{1}{2}),(2\frac{1}{2},2\frac{1}{2}),(3\frac{1}{2},2\frac{1}{2})$. Next compute $f(\frac{1}{2},\frac{1}{2})+f(1\frac{1}{2},\frac{1}{2})+f(2\frac{1}{2},\frac{1}{2})+f(3\frac{1}{2},\frac{1}{2})+f(\frac{1}{2},1\frac{1}{2})+f(1\frac{1}{2},1\frac{1}{2})+f(2\frac{1}{2},1\frac{1}{2})+f(3\frac{1}{2},1\frac{1}{2})+f(\frac{1}{2},2\frac{1}{2})+f(1\frac{1}{2},2\frac{1}{2})+f(2\frac{1}{2},2\frac{1}{2})+f(3\frac{1}{2},2\frac{1}{2})$. We get 138.

b. This represents the volume of the box. That's pretty important for the box purposes.

c. Funny that they picked 138. The exact average value would then be $\frac{138}{12}=\frac{23}{2}$.

d. If we wanted to build a box with this same volume that was rectangular, we would use the height in c., $\frac{23}{2}$.

§11.2 #1 a. $\int\int_R (f(x)+g(y))dA = \int_a^b \left(\int_c^d f(x)+g(y)dy\right)dx = \int_a^b \left([f(x)y|_c^d+37\right)dx = \int_a^b \left(f(x)(d-c)+37\right)dx = 29(d-c)+[37x|_a^b = 29(d-c)+37(b-a)$.

b. $\int_1^4 h(x,y)dx$ is a function that produces the area, for any particular y value, of the cross section under $h(x,y)$ above the xy plane, for x values between 1 and 4.

§11.2 #12 a. $\int_1^5\int_3^6 100-4x^2-y^2dydx$.

b. $\int_1^5 (100y-4x^2y-\frac{1}{3}y^3|_3^6dx = \int_1^5 300-12x^2-63dx = \int_1^5 237-12x^2dx = 237x-4x^3|_1^5 = 237(4)-4(124) = (237-124)4 = 452$. This has strange units of °C in^2. Because T has units °C and dA has units in^2

c. $4(3)=12$ in^2

d. $\frac{452}{12}=\frac{113}{3}$°C $=37\frac{2}{3}$°C. Although the answer in b. was strange and rather impractical this number is actually a useful reasonable number to want to know.

§11.3 #1 First draw the region that we're working with. It is bounded on the left by $x=0$, on the top by $y=e$, and on the bottom/right by $y=e^x$, i.e. $x=\ln y$. The lowest y-value is $y=1$ at $x=0$. From this our new limits become $\int_1^e \left(\int_0^{\ln y}\frac{1}{lny}dx\right)dy$. So,

$$\int_1^e\left(\int_0^{\ln y}\frac{1}{\ln y}dx\right)dy = \int_1^e\left[\frac{1}{\ln y}x\right|_0^{\ln y}dy = \int_1^e\frac{1}{\ln y}\ln y dy = \int_1^e dy = e-1$$

All much easier than trying to integrate $\frac{1}{\ln y}$.

§11.3 #13 a. $\int_{-2}^0\int_0^{\sqrt{4-x^2}} p(x,y)dydx$.

b. $\int_{\frac{1-\sqrt{5}}{2}}^{\frac{1+\sqrt{5}}{2}}\int_{x^2}^{x+1} p(x,y)dydx$.

c. $\int_1^2\int_x^{2x-1} p(x,y)dydx$.

d. $\int_0^4\int_0^{\sqrt{y}} p(x,y)dxdy = \int_0^2\int_{x^2}^4 p(x,y)dydx$. Either one is fine here. In all of these I haven't made the $p(x,y)$ substitutions because I think they are silly. Surely they don't matter for setup. Also, I haven't drawn the regions. Please ask me if there are questions about them. I will happily draw for you.

§11.4 #1 The top line passes thru $(0,1)$ and (a,b), so has equation $y=\frac{b-1}{a}x+1$ and the bottom line passes thru $(0,-1)$ instead so it $y=\frac{b+1}{a}x-1$. To find centre of mass we compute $\overline{x}=\frac{\int\int_R xdA}{\int\int_R 1dA}$ and $\overline{y}=\frac{\int\int_R ydA}{\int\int_R 1dA}$. The denominators are merely the area of the triangle, which could be computed by the integral, or more easily using basic geometry. The base is 2, the height is a, so the area is a. The numerators require actually computing the integral. Here's the set-up:

$$\int\int_R x dA = \int_0^a \left(\int_{\frac{b+1}{a}x-1}^{\frac{b-1}{a}x+1} x dy \right) dx = \int_0^a \left(x \left(-\frac{b+1}{a}x + 1 + \frac{b-1}{a}x + 1 \right) \right) dx = \int_0^a \left(2x - \frac{2}{a}x^2 \right) dx$$

$$= \left[x^2 - \frac{2}{3a}x^3 \right|_0^a = a^2 - \frac{2}{3a}a^3 = \frac{a^2}{3}$$

And now divide by the area, a, to get $\frac{a}{3}$ as claimed. The y-coordinate should work out about the same. Let's see:

$$\int\int_R y dA = \int_0^a \left(\int_{\frac{b+1}{a}x-1}^{\frac{b-1}{a}x+1} y dy \right) dx = \int_0^a \left[\frac{y^2}{2} \right|_{\frac{b+1}{a}x-1}^{\frac{b-1}{a}x+1} dx = \int_0^a \left(\frac{-2b}{a^2}(x^2 - ax) \right) dx$$

$$= \frac{-2b}{a^2} \left[\frac{x^3}{3} - \frac{ax^2}{2} \right|_0^a = \frac{-2b}{a^2} \frac{-a^3}{6} = \frac{ab}{3}$$

And now divide by the area, still a, to get $\frac{b}{3}$ as claimed.

Is this the average of the coordinates? Short answer - yes. Check: for x-coordinates: $\frac{0+0+a}{3} = \frac{a}{3}$, for y-coordinates: $\frac{1+(-1)+b}{3} = \frac{b}{3}$.

§11.4 #13 a. $\int_0^5 (\int_0^5 \frac{1}{8}e^{-x/4-y/2}dx)dy = \int_0^5 (-\frac{1}{2}e^{-x/4-y/2}|_0^5)dy$ $= \int_0^5 (\frac{1}{2}(e^{-y/2} - e^{-5/4-y/2})dy = (e^{-5/4-y/2} - e^{-y/2}|_0^5 = e^{-15/4} - e^{-5/2} - e^{-5/4} + 1 \simeq 0.654928$ (which is happily between 0 and 1, so could be a probability. Please note the first number is the exact probability, not the decimal approximation.)

b. $\int_0^{10} (\int_0^{10-y} \frac{1}{8}e^{-x/4-y/2}dx)dy$. Notice that $x + y \leq 10$ is reflected in the x upper limit of $10 - y$.

c. This is tricky, because there are some hidden limits here. If we're not careful we try: $\int_0^{20} (\int_{10-x}^{20-x} \frac{1}{8}e^{-x/4-y/2}dx)dy$, but that allows x to be negative if y is greater than 10. How do we work around this? I think we'll need two integrals, and to be careful with them. One is like before, but concerns x between 0 and 10, only, to avoid negatives: $\int_0^{10} (\int_{10-x}^{20-x} \frac{1}{8}e^{-x/4-y/2}dx)dy$. Now we just need x between 10 and 20. Notice, however, if x is between 10 and 20, that y is between 0 and 10, so we can use the same setup, just switch x and y. This is subtle, and you might not even notice the difference, but therefore the second part is $\int_0^{10} (\int_{10-y}^{20-y} \frac{1}{8}e^{-x/4-y/2}dy)dx$. Notice that this is not exactly the same as the first computation because the integrand is not symmetric. So, this is not two times the first answer, but it is the sum of them both: $\int_0^{10} (\int_{10-x}^{20-x} \frac{1}{8}e^{-x/4-y/2}dx)dy + \int_0^{10} (\int_{10-y}^{20-y} \frac{1}{8}e^{-x/4-y/2}dy)dx$.