We want to know when $M = 0.9e^{-0.1M}M + 1$, i.e. when $0 = 0.9e^{-0.1M}M + 1 - M$. So, we try intermediate value theorem computing the right side, with $M = 0$, we get an output of 1. You chose $M = 4$ next and we get an output of -0.586848, so the zero is between 0 and 4. To get closer we'll run the updating function once $0.9e^{-0.1*4}4+1 = 3.41315$. Ok, now let's switch to Newton's method. We're trying to find where $0 = 0.9e^{-0.1M}M + 1 - M$. The right hand side is our function to use, let's call it $f(M) = 0.9e^{-0.1M}M + 1 - M$. We'll need a derivative, $f'(M) = 0.9e^{-0.1M} - 0.09Me^{-0.1M} - 1$. And now we compute $3.41315 - f(3.41315)/f'(3.41315) = 3.01614$ and then $3.01614 - f(3.01614)/f'(3.01614) = 3.00042$, finally $3.00042 - f(3.00042)/f'(3.01614) = 3.00039$, which has oddly stabilised to 3.000, in fact to 5 places the answer is 3.00039, which is oddly close to 3, but it's not 3 exactly.

Mathematics 228 PS2 Solutions

4.1 20

Question: Is the cheetah in or out of the jungle? The book isn't clear. I'll do it both ways. Suppose the cheetah is in the savanna.

$\frac{dx}{dt} = v(t) = e^t$

The graph should look exponential, and be 1 at 0.

We integrate to find $x = e^t + C$. If we call 0 the edge of the jungle then $1 = x(0) = e^0 + C = 1 + C$, so $C = 0$. Thereore $x(t) = e^t$, which looks identical to the first graph, and we want to know when $e^t = 200$, at $\ln(200) \approx 5.2983$ seconds later.

Suppose the cheetah is in the jungle. Things are the same until $-1 = x(0) = e^0 + C = 1 + C$, so $C = -2$. Thereore $x(t) = e^t - 2$, which looks like the other graphs shifted down two, and we want to know when $e^t - 2 = 200$, at $\ln(202) \approx 5.3083$ seconds later.

4.1 28. We now use Euler's method with step 1. (I'll do this for cheetah in the savanna)

$\hat{x}_0(t) = 1t + 1$, So we approximate $x(1) = 2$

$\hat{x}_1(t) = e(t - 1) + 2$, so we approximate $x(2) = e + 2 \approx 4.718$

$\hat{x}_2(t) = e^2(t - 2) + e + 2$, so we approximate $x(3) = e^2 + e + 2 \approx 12.107$

This pattern continues and our approximation $x(5) = e^4 + e^3 + e^2 + e + 2 \approx 86.791$

The actual value is $e^5 \approx 148.413$

Since the growth is exponential, Euler is having a difficult time keeping up.

4.2 26 $\frac{dM}{dt} = 2\frac{g}{\sqrt{day}}\frac{1}{\sqrt{t}}$

Integrating produces $M = \frac{g}{\sqrt{day}}4\sqrt{t} + C$. Since M(0) = 5, C = 5, so $M(t) = \frac{g}{\sqrt{day}}4\sqrt{t} + 5$. Graphs could be on a scale of t [0,10]. M is increasing, but less and less so as $\frac{dM}{dt}$ is decreasing.

4.2 34

time	0	1	2	3	4
velocity	1	3	6	10	15

Use Euler's method with step 1 to estimate the position at t = 4 if it starts at 10.

$\hat{p}_0(t) = 1t + 10$, $\hat{p}_0(1) = 11$

$\hat{p}_1(t) = 3(t - 1) + 11$, $\hat{p}_1(2) = 14$

$\hat{p}_2(t) = 6(t - 2) + 14$, $\hat{p}_2(3) = 20$

$\hat{p}_3(t) = 10(t - 3) + 20$, $\hat{p}_3(4) = 30$

We know $v(t) = \frac{1}{2}t^2 + at + b$

because v(0) =1, b = 1.

$3 = v(1) = \frac{1}{2} + a + 1$, so $a = \frac{3}{2}$, so $v(t) = \frac{1}{2}t^2 + \frac{3}{2}t + 1$

We may check that v(2) = 6, v(3) = 10 and v(4) = 15

The exact position is $p(t) = \frac{1}{6}t^3 + \frac{3}{4}t^2 + t + 10$. Note: $p(4) = \frac{110}{3} \approx 36.67$

Euler's method is again having a little trouble keeping up with the increase.

You should have a graph of your Euler's points and a graph of p(t). The window [0,5]x[0,100] looks good to me.

4.3 36 $\frac{dW}{dt} = (4t - t^2)e^{-3t}$ $W(0) = 0$.

When is $\frac{dW}{dt}$ largest? We use calc I: $\frac{d^2W}{dt^2} = -3(4t - t^2)e^{-3t} + (4 - 2t)e^{-3t} = e^{-3t}(3t^2 - 14t + 4) = 0$ if $t = \frac{14 \pm \sqrt{148}}{6} \approx 4.3609,\ 0.30574$

At each of those times $\frac{dW}{dt} = -0.00000327$ and 0.4513736, respectively Clearly the second is larger.

Find W(2). This is done by integration.

$W(2) = W(0) + \int_0^2 (4t - t^2)e^{-3t}dt$

We need integration by parts.

Let $f = 4t - t^2$ and $g' = e^{-3t}dt$

$f' = 4 - 2tdt$ and $g = -\frac{1}{3}e^{-3t}$ so we have

$= 0 + (4t - t^2)\left(-\frac{1}{3}e^{-3t}\right) - \int_0^2 (4 - 2t)\left(-\frac{1}{3}e^{-3t}\right)dt$

$= (4t - t^2)\left(-\frac{1}{3}e^{-3t}\right) + \frac{2}{3}\int_0^2 (2 - t)(e^{-3t})dt$

Hm, we need integration by parts *again.*

Let $f = 2 - t$ and $g' = e^{-3t}dt$

$f' = -dt$ and $g = g = -\frac{1}{3}e^{-3t}$ so we have

$= (4t - t^2)\left(-\frac{1}{3}e^{-3t}\right) + \frac{2}{3}\left((2 - t)\left(-\frac{1}{3}e^{-3t}\right) - \int_0^2 \frac{1}{3}e^{-3t}dt\right)$

$= \left[(4t - t^2)\left(-\frac{1}{3}e^{-3t}\right) + \frac{2}{3}\left((2 - t)\left(-\frac{1}{3}e^{-3t}\right) + \frac{1}{9}e^{-3t}\right)\right|_0^2 = \frac{10 - 34e^{-6}}{27} \approx 0.36725$

If it always grew at the maximum rate it would be 0.90274716 instead. This is .535498 more or 2.45813 times as large. I don't know what units are used in this question.

5.1 50 $\frac{dp}{dt} = 0.5p(p - 1)$

$\hat{p}_0(t) = -0.12t + 0.6$, $\hat{p}_0(0.25) = .57$

$\hat{p}_{0.25}(t) = -0.12255(t - 0.25) + 0.57$, $\hat{p}_{0.25}(0.5) = 0.5393625$

$\hat{p}_{0.5}(t) = -0.124225296797(t - 0.5) + 0.5393625$, $\hat{p}_{0.5}(0.75) = 0.508306175801$

$\hat{p}_{0.75}(t) = -0.124965503722(t - 0.75) + 0.508306175801$,

$\hat{p}_{0.75}(1) = 0.47764799871$

The exact solution is $p(t) = \dfrac{0.6e^{2.5t}}{0.6e^{2.5t} + 0.4e^{3t}}$ $p(1) \approx 0.476383862223$

5.2 42 $\frac{dV}{dt} = a_1V^{2/3} - a_2V^{3/4}$

V is cm^3

$\frac{dV}{dt}$ is cm^3/day

$V^{2/3}$ is cm^2

a_1 is cm/day

$V^{3/4}$ is $cm^{9/4}$

so a_2 is $cm^{3/4}$/day (units like this are typical for equations made from data observations).

$a_1V^{2/3} - a_2V^{3/4} = V^{2/3}(a_1 - a_2V^{1/12}) = 0$. Either $V = 0$ or $V = \left(\frac{a_1}{a_2}\right)^{12}$

If a_1 decreases the creature gets less energy intake so grows to a smaller size. If a_2 decreases the creatures uses less energy, so grows to a larger size.

5.3 42 $\frac{dI}{dt} = \alpha I(1 - I) - I = \alpha I - \alpha I^2 - I = \alpha I\left(1 - \frac{1}{\alpha} - I\right)$

This rate is zero when $I = 0$ or when $I = 1 - \frac{1}{\alpha}$

We wish to check the stability of both of these equilibria so we take

$\frac{d}{dI}\frac{dI}{dt} = \alpha - 2\alpha I - 1$

$\frac{d}{dI}\frac{dI}{dt}(I = 0) = \alpha - 1 < 0$ stable if $\alpha < 0$

$\frac{d}{dI}\frac{dI}{dt}\left(I = 1 - \frac{1}{\alpha}\right) = \alpha - 2\alpha + 2 - 1 = 1 - \alpha < 0$ stable if $\alpha > 0$

The graph will be on an in-class handout.

5.4 31-38

$\frac{dp}{dt} = p(1 - p)$

$\frac{dp}{p(1 - p)} = dt$

Note $\frac{1}{p} + \frac{1}{1 - p} = \frac{p + 1 - p}{p(1 - p)} = \frac{1}{p(1 - p)}$

So we have

$\int\left(\frac{1}{p} + \frac{1}{1 - p}\right)dp = \int dt$

Integrating both sides we get

$\ln(p) - \ln(1 - p) = t + c$

$\ln\left(\frac{p}{1 - p}\right) = t + c$

So $\frac{p}{1 - p} = Ke^t$ and $p = (1 - p)Ke^t = Ke^t - Kpe^t$ and $p = \frac{Ke^t}{1 + Ke^t}$

If $p(0) = 0.01$ we have $0.01 = \frac{K}{1 + K}$ so $K = \frac{1}{99}$, $\lim_{t\to\infty} \frac{\frac{1}{99}e^t}{1 + \frac{1}{99}e^t} = 1$

If $p(0) = 0.5$ we have $0.5 = \frac{K}{1 + K}$ so $K = 1$, $\lim_{t\to\infty} \frac{e^t}{1 + e^t} = 1$

5.5 34 $\frac{dI}{dt} = \alpha IS - \mu I$; $\frac{dS}{dt} = -\alpha IS + \frac{2}{3}\mu I$

5.5 38 $\frac{dI}{dt} = \alpha IS - \mu I$; $\frac{dS}{dt} = -\alpha IS + \mu I + b(I + S)$

5.6 32a 2β b β c $\frac{dA_1}{dt} = \beta C_2 - 2\beta C_1$, $\frac{dA_2}{dt} = 2\beta C_1 - \beta C_2$

d. $\frac{dC_1}{dt} = \frac{1}{V_1}(\beta C_2 - 2\beta C_1)$

$\frac{dC_2}{dt} = \frac{1}{V_2}(2\beta C_1 - \beta C_2)$

The nullcline for C_1 is given by $\beta(C_2 - 2C_1) = 0$, i.e. $C_2 = 2C_1$.

The nullcline for C_2 is given by $\beta(2C_1 - C_2) = 0$, i.e. $C_2 = 2C_1$

Since the nullclines are identical, any concentration that meets this ratio will be stable, it is only a matter how much total chemical is around.

5.6 40 find nullclines and equilibria of 5.5 38

$\frac{dI}{dt} = 2IS - I$

$\frac{dS}{dt} = -2IS + I + (I + S) = -2IS + 2I + S$

The nullclines of I: $2IS - I = 0$ are $I = 0$ and $S = \frac{1}{2}$

The nullcline of S: $-2IS + 2I + S = 0$ is $S = 1 + \frac{1}{2I - 1} = \frac{2I}{2I - 1}$

To find the equilbria we find the intersections. On the S nullcline if $I = 0$, we have $S = 0$, our old (0,0) equilibrium. If $S = \frac{1}{2}$, we get $I = -\frac{1}{2}$. This is not biologically possible, so the only equilibrium is at (0, 0).

5.7.36 add direction arrows to the previous

We'll compute the derivatives at some points (I'm using (I, S) coordinates, so I is the first coordinate)

I	S	$\frac{dI}{dt}$	$\frac{dS}{dt}$
0	0	0	0
0	1	0	1
1	0	−1	2
1	1	1	1
0	2	0	2
2	0	−2	4
2	2	6	−2
1	2	3	0
2	1	2	1
0	0.5	0	0.5
0.5	0.5	0	1

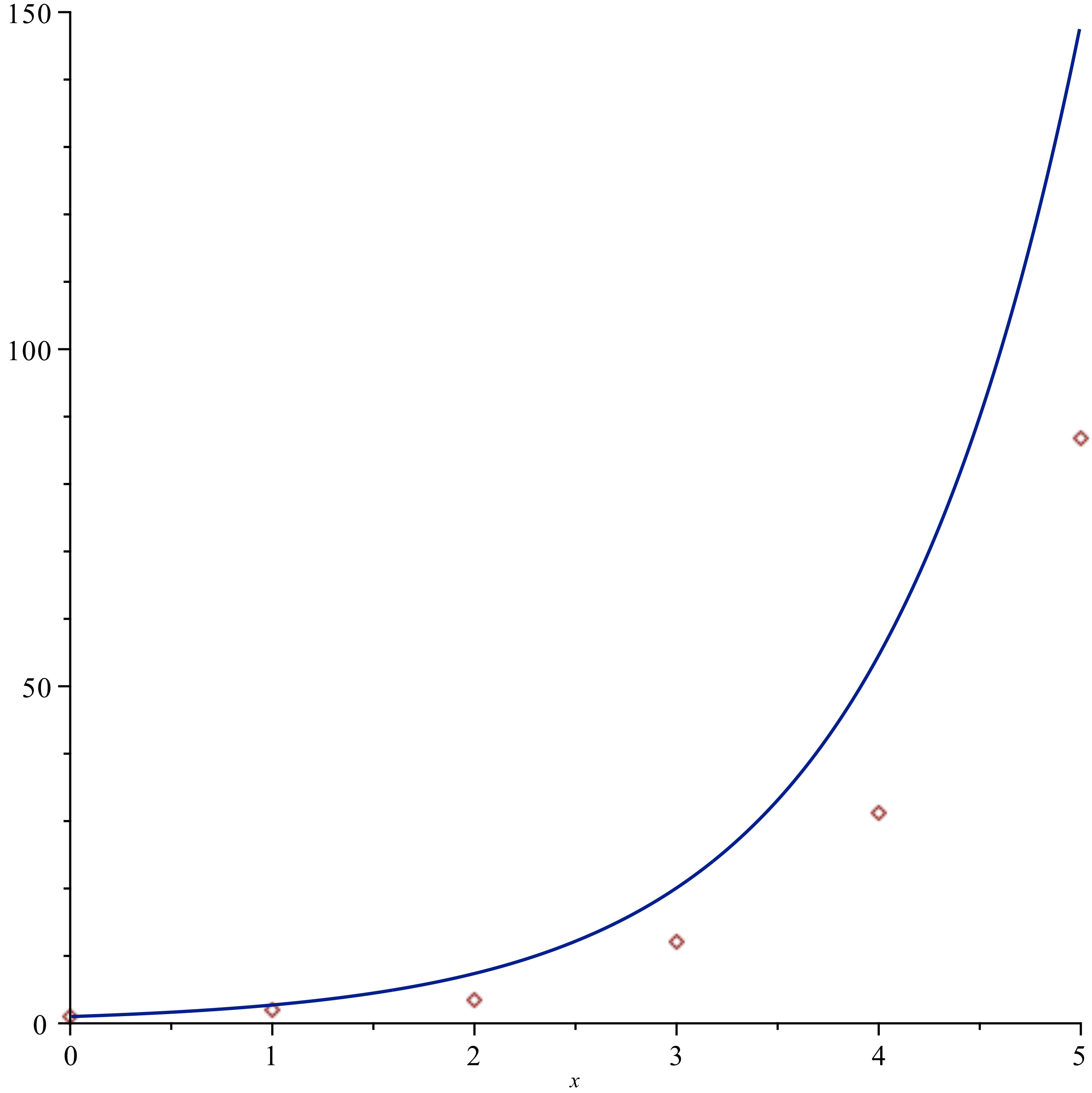
5.7.44 draw a trajectory starting at (0.5, 0.5) for the previous.

Follow the arrrows starting at (0.5, 0.5). Both are drawn on the class handout, which will also include nullclines and the equilibrium.

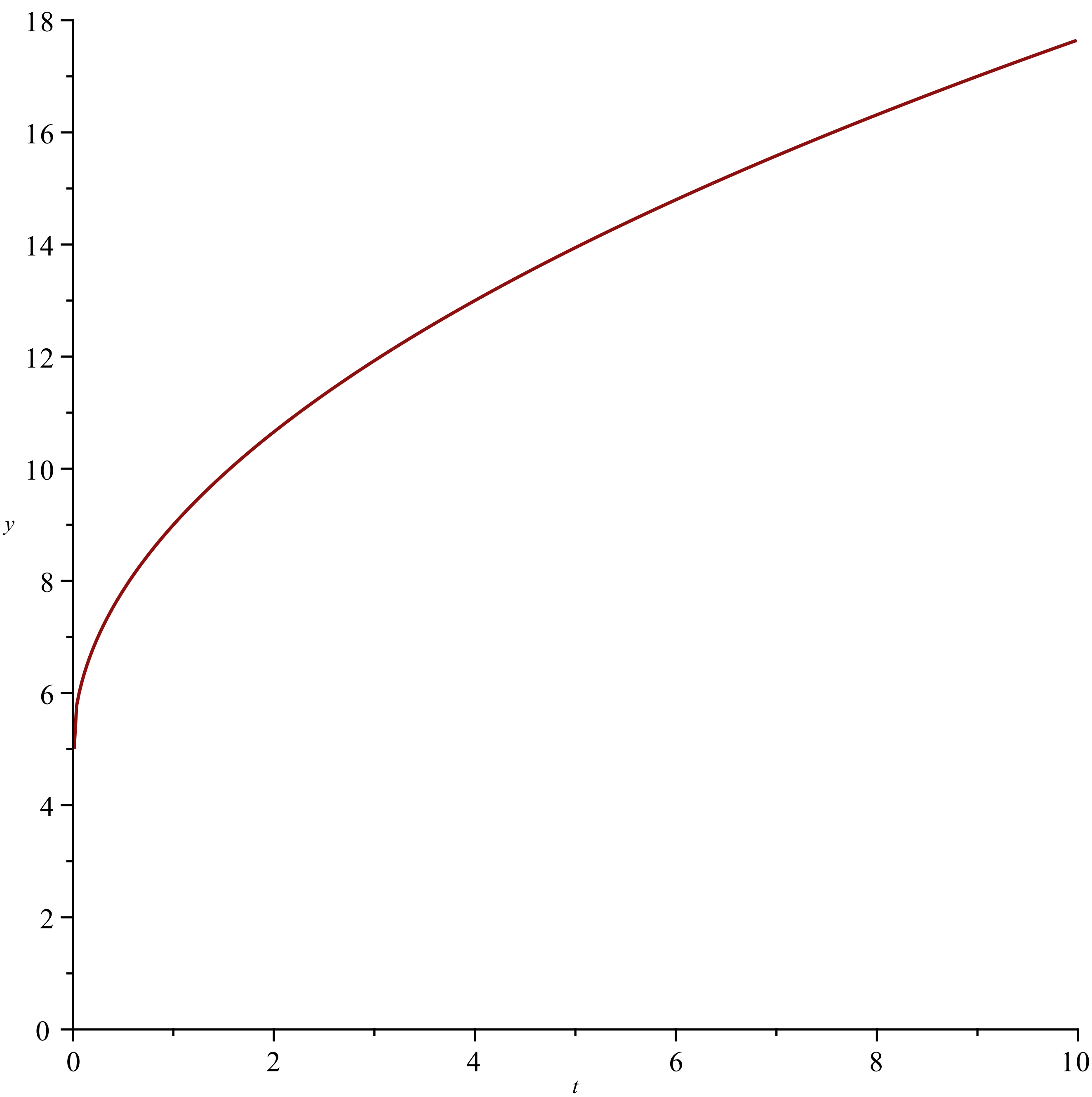
> *with*(*plots*, *textplot*, *display*);

$$[textplot, display] \quad \textbf{(1)}$$

> *plot1* := *plot*([[0, 1], [1, 2], [2, 3.4718], [3, 12.107], [4, 31.193], [5, 86.791]], *style* = *POINT*) :
plot2 := *plot*(exp(*x*), *x* = 0 ..5) :
display(*plot1*, *plot2*, *view* = [0 ..5, 0 ..150]);



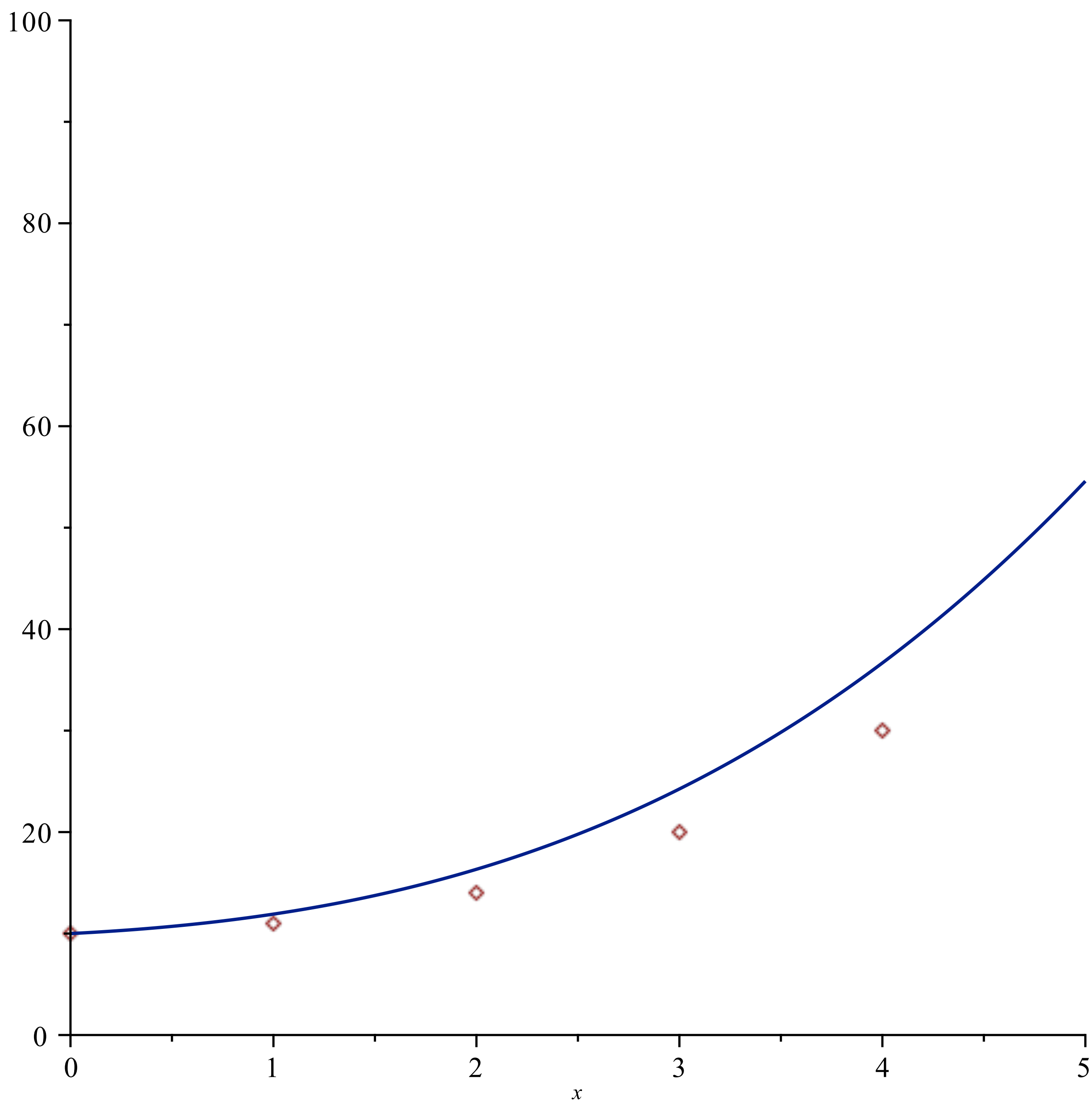
> *plot*(4·sqrt(*t*) + 5, *t* = 0 ..10, *y* = 0 ..18);



> $plot1 := plot([[0, 10], [1, 11], [2, 14], [3, 20], [4, 30]], style = POINT)$:

$plot2 := plot\left(\frac{1}{6}x^3 + \frac{3}{4}x^2 + x + 10, x = 0..5\right)$:

$display(plot1, plot2, view = [0..5, 0..100])$;


100
80
60
40
20
0
0
1
2
3
4
5
x

>