233 Problem Set 1 Solutions

1.2.1. I asked you to show all your work, and I will do the same because I know that you're learning. There's a lot to do here. I'm not copying the equations, but I am doing all the rest.

a. Here's the augmented matrix:

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 3 \\ -2 & 2 & 6 \end{bmatrix}$$

For my first move I will switch the first two rows:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ -2 & 2 & 6 \end{bmatrix}$$

Next I will multiply it by negative and positive two and add to the second and third rows respectively:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 6 & 12 \end{bmatrix}$$

At this point I'm tempted to just notice that the last two is a multiple of the second and so will cancel, but I'll be more systematic. So, let's divide the middle row by -3 (unless you say we need to multiply in which case I will say "fine, multiply by $-\frac{1}{3}$"):

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 6 & 12 \end{bmatrix}$$

Now, add -2 times this row to the first, and as expected, cancel the last by adding -6 times the middle:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

And we translate back to equations to find $x = -1$ and $y = 2$ with the true but not helpful $0 = 0$. For anything that matters, it's good to check. (I checked my work, and I was wrong.). Oh, the two variables are both free.

Ok, that's a good start.

b. We start again:

$$\begin{bmatrix} -1 & 2 & 1 & 2 \\ 3 & 0 & 2 & -1 \\ -1 & -1 & 1 & 2 \end{bmatrix}$$

I think the last row will be just a bit easier so I'm going to multiply it by -1 and move it to the top.

$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ -1 & 2 & 1 & 2 \\ 3 & 0 & 2 & -1 \end{bmatrix}$$

Now I will add this row to the second, and add -3 times the first row to the third.

$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 3 & 0 & 0 \\ 0 & -3 & 5 & 5 \end{bmatrix}$$

Oh, that middle row is going to be easy. It's so easy, I'm going to do it all at once. Ok, so, divide by 3, then because it's just that one there, when we combine it with other rows, it easily makes zeroes and changes no other values.

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 5 \end{bmatrix}$$

Next we divide the last row by 5, and we're almost done.

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

And now we add the last row to the first row, and we're done:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

So, $x = -1$, $y = 0$, $z = 1$. Check it, and ta da. Oh, the three variables are all basic.

c. Now, in 4d, but that doesn't scare us:

$$\begin{bmatrix} 1 & 2 & -5 & -1 & -3 \\ -2 & -2 & 6 & -2 & 4 \\ 1 & 0 & -1 & 9 & -7 \\ 0 & -1 & 2 & -1 & 4 \end{bmatrix}$$

Again, I'm moving the third row to the first, I like the numbers. In fact, I'll just shuffle a bit, putting the last two first.

$$\begin{bmatrix} 1 & 0 & -1 & 9 & -7 \\ 0 & -1 & 2 & -1 & 4 \\ 1 & 2 & -5 & -1 & -3 \\ -2 & -2 & 6 & -2 & 4 \end{bmatrix}$$

Now subtract the first from the third, and add twice the first to the fourth to get:

$$\begin{bmatrix} 1 & 0 & -1 & 9 & -7 \\ 0 & -1 & 2 & -1 & 4 \\ 0 & 2 & -4 & -10 & 4 \\ 0 & -2 & 4 & 16 & -10 \end{bmatrix}$$

The next natural move is to multiply the second row by -1. I notice that the last two rows are all even numbers, so I'm going to also make this easier by just dividing everything by two in those rows:

$$\begin{bmatrix} 1 & 0 & -1 & 9 & -7 \\ 0 & 1 & -2 & 1 & -4 \\ 0 & 1 & -2 & -5 & 2 \\ 0 & -1 & 2 & 8 & -5 \end{bmatrix}$$

Next subtract the first from the second, and then add the first to the third:

$$\begin{bmatrix} 1 & 0 & -1 & 9 & -7 \\ 0 & 1 & -2 & 1 & -4 \\ 0 & 0 & 0 & -6 & 6 \\ 0 & 0 & 0 & 9 & -9 \end{bmatrix}$$

Now let's divide the penultimate by -6, and last by 9 to get:

$$\begin{bmatrix} 1 & 0 & -1 & 9 & -7 \\ 0 & 1 & -2 & 1 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Next subtract the third row from the last and second, and for the first row subtract nine times the third all to get:

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 2 \\ 0 & 1 & -2 & 0 & -3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

And now we translate out to get a parametrisation. The last one is easy, $x_4 = -1$. The other two are less nice: $x_1 - x_3 = 2$ and $x_2 - 2x_3 = -3$. But we can clean it up mostly nicely. x_1, x_2, and x_4 are basic. x_3 is free this time, and we can solve for the others in terms of it, $x_1 = x_3 + 2$, and $x_2 = 2x_3 - 3$ and, of course $x_4 = -1$.

1.2.3. a. reduced row echelon with five equation and three variables with exactly one solution (remember the matrix is augmented because it represents a linear system):

$$\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -12 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There's not really a lot of options here, just changing the constants in the first the rows.

b. three equations and three variables and no solution.

$$\begin{bmatrix} 1 & 0 & -7 & 0 \\ 0 & 1 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The only other option here is something like

$$\begin{bmatrix} 1 & 2.7 & 0 & 0 \\ 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c. three equations and five variables with infinitely many solution.

$$\begin{bmatrix} 1 & -7.9 & 0 & 0 & 687 & 0 \\ 0 & 0 & 1 & 0 & 7 & -1.5 \\ 0 & 0 & 0 & 1 & 34 & 7.25 \end{bmatrix}$$

There's a lot of options here.

d. three equations and four variables with exactly one solution. This is not possible. You would need four pivots, but there are only three equations.

e. four equations, four variables one solution. There's basically no options here, aside from the constants:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -8/7 \\ 0 & 1 & 0 & 0 & -7.3 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 789 \end{bmatrix}$$

1.2.5.

a. What does a "Z-row" tell us? Nothing! It tells us that 0=0, which is true, but not informative in the least.

b. How can you determine if it has no solutions (is inconsistent)? If there's a row that says 0=1. It will always be 0=1, because of our row reduction rules for echelon form.

c. How can you tell if it has infinitely many solutions? Well, not by the Z-row, in a. Instead, by there being a variable that does not have a leading one. That makes it a free variable, which, being free, can take any value, hence infinitely many solutions.

d. If more variables than equations, then there are not enough equations to make all the variables basic. As long as there is not a contradiction, there therefore will be free variables, hence infinitely many solutions.

1.3.3. This is mostly to demonstrate that we don't want do this work by hand. Here we go with some equations:

$$4T_1 = 10 + 15 + T_2 + T_4, 4T_2 = 20 + T_1 + T_3 + T_5, 4T_3 = 25 + T_2 + 30 + T_6$$
$$4T_4 = 10 + T_1 + T_5 + 10, 4T_5 = T_2 + T_4 + T_6 + 20, 4T_6 = T_3 + T_5 + 40 + 30$$

This should look better in a matrix:

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 25 \\ -1 & 4 & -1 & 0 & -1 & 0 & 20 \\ 0 & -1 & 4 & 0 & 0 & -1 & 55 \\ -1 & 0 & 0 & 4 & -1 & 0 & 20 \\ 0 & -1 & 0 & -1 & 4 & -1 & 20 \\ 0 & 0 & -1 & 0 & -1 & 4 & 70 \end{bmatrix}$$

There's lots of neat patterns that match the patterns that we started with. Anyway, now we set sage to work. Ok, I agree with the text that fractions with denominators of 483 are probably not so useful. Here's the decimal approximations, in order: $(14.86, 20.50, 26.19, 13.95, 20.93, 29.28)$. I'm glad I didn't do that by hand.

1.3.5. a. Hydrogen equation $2x = 2z$, Oxygen equation $2y = z$. As a matrix this is then

$$\begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix}$$

I don't see how sage was helpful here, but row reducing just merely dividing the two rows by two to get:

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \end{bmatrix}.$$

Which says, not shockingly that there are as many H_2 molecules as water, and half as many O_2 as water. That's pretty obvious. There are infinitely many solutions either practically because we can make as much water as we want or mathematically because there are two equations and three unknowns. The smallest whole number is for one O_2 molecule, so $(x, y, z) = (2, 1, 2)$.

b. Ok, this is why we want sage, got it.
Potassium: $x_1 = 2x_5$
Manganese: $x_1 + x_2 = x_4$
Oxygen: $4x_1 + 4x_2 + x_3 = 2x_4 + 4x_5 + 4x_6$
Sulfur: $x_2 = x_5 + x_6$
Hydrogen: $2x_3 = 2x_6$
So, now to put this into a matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 4 & 4 & 1 & -2 & -4 & -4 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 & -2 & 0 \end{bmatrix}$$

Sage tells me this reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -3/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -5/2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1/2 & 0 \end{bmatrix}$$

To find the smallest values we start with $x_6 = 2$ and work up from there, just to get the /2 fractions to go away. In order we then get $(2, 3, 2, 5, 1, 2)$.

1.4.7. Consider this matrix:

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 4 \\ a & b & c & 9 \end{bmatrix}$$

a. How do we make it inconsistent? The obvious answer is $a = b = c = 0$. It's important that this is not the only one. Here are two other pretty obvious ones: $(1, 2, 3)$, $(4, 5, 6)$, they already equal 1 and 4, they can't equal 9. Here's another one, that previews one of the other parts, multiply the first row by 10, add to the second row, but ... don't get 14, because there's a nine there. That would be $(14, 25, 36)$.

b. How to make it have a unique solution? I don't know about you, but I did this part last. This is the hard one. The best answer I can give at this point in the course is this is those that are not the two other cases. We will learn more about this later. I will try to remember this question and come back it then. We could write the precise condition that (a, b, c) need to solve. For you? I'd say this a good place to play with sage. Or pick values and row reduce by hand. Here are some that I found: my first guess was $(0, 0, 9)$ which gives a nice simple solution. In fact, this generalises quite a bit $(0, 0, c)$ works for any nonzero c, same for $(a, 0, 0)$, and $(0, b, 0)$. I'll take all those. And, I will stop there to give more space for you below. It actually turns out that this is the most common situation. If you pick three arbitrary numbers they will almost always give you a unique solution. Unless the particular point (a, b, c) lies on a particular plane in three dimensions. Again, I hope to remember this when we get there.

c. This is the opposite of part a. Probably the most obvious answer is to multiply the first row by 9 to get $(9, 18, 27)$. I like multiplying the first row by 5 and adding the second to get $(9, 15, 21)$. Similarly we can add one of the first row and twice the second to get $(9, 12, 15)$. Note that in all cases $a = 9$, that is forced because the first and last columns are otherwise the same. Each of these produces a row that will cancel in row reduction.

There's lots of different answers to each of these. This could be a fun game. If you email me an example that is different from mine (for each of part of the problem) and from anyone else's I'll give +1 on this Problem Set. And, because I know they were reading solutions ... +2 for the *first* person to send to me.

1.4.9. Before we do this, I want to acknowledge the exception to this. If the two points are vertical, there is a line, but it isn't a function. If the four points were vertical this problem would persist. If the four points are on a line, then $c = 0$ and $d = 0$, which I guess is a degenerate cubic. The four points are $(0, 2)$, $(1, 3)$, $(2, 0)$, and $(3, 1)$.

a. So, the equations are $2 = a + 0b + 0c + 0d$, $3 = a + b + c + d$, $0 = a + 2b + 4c + 8d$ and $1 = a + 3b + 9c + 27d$.
b. This is the matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 8 & 0 \\ 1 & 3 & 9 & 27 & 1 \end{bmatrix}$$

Hey, sage, help me out ... thanks, those values look yucky, I'm glad you found them and not me $p(x) = 2 + \frac{17}{3}x - 6x^2 + \frac{4}{3}x^3$.

c. It was fun to plot in sage. I hope you did. Here's the command: "`plot(2 + 17/3*x - 6*x^2 + 4/3*x^3,xmin=-1,xmax=4)`". [Aside: wow, I've been writing in LaTeX for something like 30 years, it's really difficult to get it to type the ^character.] Anyway, the graph looks to go through the desired points.

d. What about a quadratic, feels unlikely. The new matrix is: This is the matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 0 \\ 1 & 3 & 9 & 1 \end{bmatrix}$$

Unsurprisingly Sage agrees with me, as this row reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Which is an important and great looking matrix, but this is the augmented matrix of a clear contradiction. So, there's no quadratic through these four points. That is visually apparent from trying to hand-draw a polynomial using the four points.

e. Ok, we're moving up to degree four. Fine. The new matrix is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 8 & 16 & 0 \\ 1 & 3 & 9 & 27 & 81 & 1 \end{bmatrix}$$

I'm not shocked that there are infinitely many choices. If $e = 0$, we return to the cubic from before. This is the matrix Sage tells me.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 6 & \frac{17}{3} \\ 0 & 0 & 1 & 0 & -11 & -6 \\ 0 & 0 & 0 & 1 & 6 & \frac{4}{3} \end{bmatrix}$$

Thanks Sage!

f. This problem hasn't been hard, but tedious to typeset. The results aren't surprising. Two points (usually) give a unique line, four points (usually) give a unique cubic (degree 3). You may (I hope) know that three points (usually) give a unique parabola. By analogy and from what we saw with the linear algebra, ten points usually gives a unique ninth degree polynomial. The one exception is when at least two of the ten points are vertical.

2.1.5. a. Can we express $\mathbf{b} = \begin{bmatrix} 10 \\ 1 \\ -8 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix}$?

We use this matrix to solve for a, b, c:

$$\begin{bmatrix} 2 & 0 & 4 & 10 \\ -1 & 3 & 4 & 1 \\ -2 & 1 & -2 & -8 \end{bmatrix}$$

Sage says

$$\begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

i.e. $a = 5 - 2c$, $b = 2 - 2c$. There are apparently infinitely many ways to do so, including skipping $\mathbf{v}_3$ entirely. Ok.

b. Can we express $\mathbf{b} = \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}$ as a linear combination? This is the matrix now:

$$\begin{bmatrix} 2 & 0 & 4 & 3 \\ -1 & 3 & 4 & 7 \\ -2 & 1 & -2 & 1 \end{bmatrix}$$

This time Sage says

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i.e. no we cannot, as this is inconsistent.

c. To express $\mathbf{v}_3$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$ we use:

$$\begin{bmatrix} 2 & 0 & 4 \\ -1 & 3 & 4 \\ -2 & 1 & -2 \end{bmatrix}$$

Now Sage tells us:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

which was perhaps obvious, $2\mathbf{v}_1 + 2\mathbf{v}_2 = \mathbf{v}_3$. If nothing else we should see that this is the same row reduction as in a. and b., only using the first three columns and not the fourth.

e. This is pretty simple: $a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = a\mathbf{v}_1 + b\mathbf{v}_2 + c(2\mathbf{v}_1 + 2\mathbf{v}_2) = (a + 2c)\mathbf{v}_1 + (b + 2c)\mathbf{v}_2$. I don't know that that needs explanation. That seems pretty clear.

2.1.7. a. $2\mathbf{v} = 2\mathbf{v} + 0\mathbf{w}$. This is so obvious it's almost tricky. I've got no more to say.

b. This is true. If we row reduce $[\mathbf{v}_1\mathbf{v}_2 \cdots \mathbf{v}_n\mathbf{b}]$ (which as we see from 2.1.5 is what it takes to write $\mathbf{b}$ as a linear combination of the others), we would get a matrix with pivots in every row, and none would be in the last column since we augmented to include $\mathbf{b}$.

c. This is *also* true. The row reduced coefficient matrix looks like

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which we saw before. This time the matrix is powerful as we can augment it with and $\mathbf{b}$ to get a unique solution.

d. This is *false*. The short answer is that this is what 3-dimensional means, that it takes three vectors. From the point of view of this problem, what would this say? It would require that $[\mathbf{v}_1\mathbf{v}_2\mathbf{b}]$ could always (for any $\mathbf{b}$) row reduce to a matrix having a zero row at the bottom, because this matrix is 3×3. So, the two coefficient would need to be enough to determine $\mathbf{b}$. At best we can solve the first two equations (components) for coefficients on $\mathbf{v}_1$ and $\mathbf{v}_2$, but when doing so that determines a relationship for the third equation. Therefore for most third components of $\mathbf{b}$ this relation will *not* hold. I will be interested to see how you explain this. Explanations matter.