233 Problem Set 2 Solutions

§2.2.7. a. $SA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ -3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 14 & 0 & 14 \\ -3 & 2 & 3 \end{bmatrix}$, as expected. If we wanted to multiply the third row by -3 instead, we would use $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

b. $PA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ -3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 2 & -1 \\ -3 & 2 & 3 \end{bmatrix}$, as expected. If we wanted to switch the first and third instead, we would use $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

c. $L_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ -3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -4 & 4 \\ -3 & 2 & 3 \end{bmatrix}$, as expected. If we wanted to multiply the first row by 3 and add to thie third instead, we would use $L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$.

Notice in all three cases, the matrix that does that action is the result of doing that action on the identity matrix. Think about this. Remember multiplying by the identity on the right it doesn't change the matrix on the left, so the matrix on the left needs to be the matrix that is the effect of the operation on the identity.

d. The nice thing is that we already got a start on this in c., and then we even had a hint for step two. And, there are only three steps, so ... that's not bad.

To review, and pull together, we started with

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ -3 & 2 & 3 \end{bmatrix}$$

. The first step was taken above, multiplying by $L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. That changes to the matrix, as stated above to

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -4 & 4 \\ -3 & 2 & 3 \end{bmatrix}$$

We were told the next step uses matrix $L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ which multiplies the first row by 3 and adds to the last row to get

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -4 & 4 \\ 0 & 8 & 0 \end{bmatrix}$$

And now we only have one step left. This is to multiply the second row by 2 and add to the last. This uses matrix $L_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ And it gives us

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -4 & 4 \\ 0 & 0 & 8 \end{bmatrix}$$

And, we're almost done - finally! That is our matrix $U = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -4 & 4 \\ 0 & 0 & 8 \end{bmatrix}$. To find L, we multiply our left matrices one by one, on the left to get $L = L_3 L_2 L_1$.

So, $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$.

We should check $LA = U$, so we try it with Sage, and ... hooray (although I did notice a mistake by checking).

§2.2.13. Catching up from 12, $A = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}$.

There's not much to do here, but, yes,

$$\begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

And similarly,

$$\begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.3 \\ 0.3 \end{bmatrix} = 0.3 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

b. This is kinda dull, also $\mathbf{x}_2 = A\mathbf{x}_1 = A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2)$. We have two properties of multiplication that we saw in activity 2.2.3 parts d. and e., $A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w}$, and $A(c\mathbf{v}) = cA\mathbf{v}$. Using the first one we get $A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = A(c_1\mathbf{v}_1) + A(c_2\mathbf{v}_2)$ then using the second (twice) $= c_1 A(\mathbf{v}_1) + c_2 A(\mathbf{v}_2)$ and now we can use the results in a. $= c_1(\mathbf{v}_1) + c_2(0.3)(\mathbf{v}_2)$ and we commute the constants to get $= c_1(\mathbf{v}_1) + 0.3c_2(\mathbf{v}_2)$, as desired.

c. The rest is pretty similar, each time we compute again, we multiply c_2 by 0.3, hence we get what is written. I'm not very concerned about this one.

d. $\mathbf{x}_1 = \begin{bmatrix} 500 \\ 500 \end{bmatrix}$. Find $\mathbf{x}_1 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. For this we have $\mathbf{x}_1 = \begin{bmatrix} 5 & -1 \\ 2 & 1 \end{bmatrix} \mathbf{c}$. Hence we reduce $\begin{bmatrix} 5 & -1 & 500 \\ 2 & 1 & 500 \end{bmatrix}$. So, we find $c_1 = \frac{1000}{7}$, and $c_2 = \frac{1500}{7}$.

e. $\mathbf{x}_2 = \frac{1000}{7}\mathbf{v}_1 + 0.3\frac{1500}{7}\mathbf{v}_2 = \begin{bmatrix} 650 \\ 350 \end{bmatrix}$

and $\mathbf{x}_3 = \frac{1000}{7}\mathbf{v}_1 + 0.3^2\frac{1500}{7}\mathbf{v}_2 = \begin{bmatrix} 695 \\ 305 \end{bmatrix}$

while $\mathbf{x}_4 = \frac{1000}{7}\mathbf{v}_1 + 0.3^3\frac{1500}{7}\mathbf{v}_2 = \begin{bmatrix} 708.5 \\ 291.5 \end{bmatrix}$.

f. So, what is this getting close to? That's unclear, but here's an idea ... the more we multiply by 0.3, the smaller that part gets, so this gets closer to just $c_1\mathbf{v}_1$ which is $\begin{bmatrix} \frac{5000}{7} \\ \frac{2000}{7} \end{bmatrix}$, which is more cleanly said that $\frac{5}{7}$ of the bicycles are at B and $\frac{2}{7}$ of the bicycles are at C. And ... now we know.

§2.3 9. a. There is pivot position in every row and every column, that's convenient.

b. Hence the span of the columns is all of $\mathbb{R}^{12}$.

c. If $\mathbf{c}$ is some other vector, the matrix A still reduces the same as before, so there's some different solution, but only one.

d. Because there's only one for any $\mathbf{c}$, there is only one for $\mathbf{0}$. But we know that $A\mathbf{0} = \mathbf{0}$. So, that is the entire solution space.

§2.3.11 a. To analyse the span, we can find the matrix of with the three vectors as columns and reduce. If it reduces to the identity, any vector is in the span. If it reduces to something with fewer rows, then the vectors which correspond to basic (not free) variables, can be used to span the span without the others. In this case, $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -2 & 0 & 2 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ we should have seen (I didn't) that $\mathbf{v}_2 - \mathbf{v}_1 = \mathbf{v}_3$. So, the first two vectors suffice, and the span is the plane spanned by the linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$. Not a surprise that it doesn't include all the vectors.

b. Let's see if Sage will do what is asked for us, I'm not sure how good it is at algebra with variables. We're starting with the matrix $\begin{bmatrix} 1 & 2 & 1 & a \\ 0 & 1 & 1 & b \\ -2 & 0 & 2 & c \end{bmatrix}$ Sage is unhappy with this, ok. So, we reduce this matrix by hand. We can do it. It's ok. No reason to be lazy, and there's already a good 1 and 0. And it only needs to be triangular. It takes only two steps, twice the first row added to the third, and then -4 times the first row added to the third yields $\begin{bmatrix} 1 & 2 & 1 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 0 & c+2a-4b \end{bmatrix}$. So, for the system to be consistent we need $0 = c + 2a - 4b$. So the set of all vectors $\langle a, b, c\rangle$ such that $0 = c + 2a - 4b$ is the span.

c. Geometrically the span would be a vector. But, that's not exactly the question. There would only be one basic variable and two free variables, so there would be *two* rows of zeroes. That would give *two* equations with the three variables a, b, and c; the linear system has two equations instead of one. That is how it changes the system, and that is the question.

§2.4.3. a. Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent (I'm using the definition that they have a non-zero linear combination equal to the zero vector. This is equivalent to one being a sum of the others, but the equations are easier to write this way. I won't mark wrong for using the definition used in the book, it's just more difficult to set up in general). Therefore $a\mathbf{v}_1 + b\mathbf{v}_2 = \mathbf{0}$ for some a and b, not both zero. We can then write $a\mathbf{v}_1 = -b\mathbf{v}_2$ and divide by either a or $-b$, picking one that isn't zero to find one vector as a scalar multiple of another.

b. If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an independent set of vectors, then the only way that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ is with all $c_i = 0$. Since there's no way to make a nonzero linear combination with all n vectors, there is also no way to make a nonzero linear combination with fewer of them (it would be like using zeroes for the omitted vectors), hence any nonempty subset is also independent.

c. If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an independent set of vectors comprising the columns of A, and $A\mathbf{x} = \mathbf{b}$ is inconsistent, this is equivalent to making a matrix with columns: $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \mathbf{b}$. Because the original set is independent there is a pivot in each column. Because $\mathbf{b}$ is inconsistent, if we include it as a column, we will have a pivot in a row that does not have a pivot for the first vectors, therefore, if we augment the matrix once more by including $\mathbf{0}$ we cannot find a nonzero linear combination of the vectors that yields zero, so $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \mathbf{b}\}$ is also an independent set.

§2.4.7 Ok, this feels not bad. More work for me than you, only because I need to type the matrices. Away we go.

a. 4 independent vectors in $\mathbb{R}^5$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the only possible answer here. They are in $\mathbb{R}^5$ so there are 5 rows, we need 4 pivots, and that's the only place they can be in reduced form.

b. 4 independent vectors in $\mathbb{R}^4$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This is again the only possible answer here. This is almost identical to the first one, one fewer row since only four dimensional vectors.

c. $\mathbb{R}^4$ is 4-dimensional and cannot be spanned by 3 vectors. This is impossible. I would need 4 pivots in 3 columns.

d. To do this I would need 5 pivots in three rows. This is also impossible.

e. 5 vectors whose span is $\mathbb{R}^4$: $\mathbb{R}^4$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 24 \\ 0 & 0 & 0 & 1 & 13 \end{bmatrix}.$$

Finally one that has more than one answer although not much. The only variation here is that we are free to choose anything for the last column. I admit, it would look suspicious to me if the last column were all zeroes. It's not wrong, but it's much more special than it needs to be.

§2.5.5. I hope you like that we're connecting back to our in-class activities. That feels good to me.

a. 1. $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 10 & 20 \\ 50 & 30 \\ 30 & 30 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10x_1 + 20x_2 \\ 50x_1 + 30x_2 \\ 30x_1 + 30x_2 \end{bmatrix}$. That's all of that, but it's a part of a part.

2. The first row is cakes, so there are $10x_1 + 20x_2$ cakes.

3. Bakery one works for x_1 hours and makes 10 cakes per hour, so they make $10x_1$ cakes. The other bakery makes 20 cakes/hour and works for x_2 hours, so they make $20x_2$ cakes. And, that's that.

b. This should remind you of the trucks, and of course when we saw bicycles in §2.2.

1. We want to know about those that end at P. If they begin at P and stay there that happens 0.6 of the time. If they begin at Q and end up at P, that happens 0.7 of the time. So, x_1 bicycles start at P, of those $0.6x_1$ return to P. And x_2 bicycles start at Q, of those $0.7x_2$ return to Q. So, altogether there are $0.6x_1 + 0.7x_2$ bicycles at P at the end of the day.

2. Similarly there are $0.4x_1 + 0.3x_2$ at Q.

3. Just combining the two above gives $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 0.6x_1 + 0.7x_2 \\ 0.4x_1 + 0.3x_2 \end{bmatrix}$

4. So $A = \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix}$

This question should all feel painfully slow, but I hope it clarifies the connection for anyone who was missing it.

§2.5.7. This also should sound pretty familiar.

a. I hope by now we can just write the columns to get $A = \begin{bmatrix} 20 & 30 & 0 \\ 15 & 5 & 40 \end{bmatrix}$.

b. Similarly the matrix $B = \begin{bmatrix} 5 & 6 \\ 8 & 10 \end{bmatrix}$. (As a tip: I'm looking at the output and seeing that the first row is electricity, E, and that helps me to keep it straight.)

c. We compute $A\mathbf{x} = \begin{bmatrix} 20 & 30 & 0 \\ 15 & 5 & 40 \end{bmatrix} \begin{bmatrix} 30 \\ 20 \\ 10 \end{bmatrix} = \begin{bmatrix} 1200 \\ 950 \end{bmatrix}$. That's the first question. Now we compute $B\mathbf{y}$ for the second question to get $\begin{bmatrix} 5 & 6 \\ 8 & 10 \end{bmatrix} \begin{bmatrix} 1200 \\ 950 \end{bmatrix} = \begin{bmatrix} 11700 \\ 19100 \end{bmatrix}$. To be clear that's (in the ambiguous "units") 11700 units of energy and 19100 units of labour.

d. $C = BA$, notice in part c. we did A first then B, we always work on the left with function composition. This gives $C = \begin{bmatrix} 20 & 30 & 0 \\ 15 & 5 & 40 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 8 & 10 \end{bmatrix} = \begin{bmatrix} 190 & 180 & 240 \\ 310 & 290 & 400 \end{bmatrix}$. These §2.5 questions felt like a break after the first questions. I hope that's comforting.

§2.6.3. a. To find the matrix, we want to know the effects on $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. The effects of each will give the columns respectively. I'll do the easy one first $T(\mathbf{e}_1) = \mathbf{e}_1$, it isn't moving because we're rotating around it. $T(\mathbf{e}_2) = \mathbf{e}_3$, as the y direction moves to the z direction. Finally $T(\mathbf{e}_3) = -\mathbf{e}_2$, as z moves to y, but negative. We can put this together into matrix $X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$. That wasn't too bad.

b. Ok, now we do it around y. I'll do the easy one first $T(\mathbf{e}_2) = \mathbf{e}_2$, it isn't moving because we're rotating around it. $T(\mathbf{e}_3) = \mathbf{e}_1$, as the z direction moves to the x direction. Finally $T(\mathbf{e}_1) = -\mathbf{e}_3$, as x moves to z, but negative. We can put this together into matrix $Y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$.

c. One more time, around z. I'll do the easy one first $T(\mathbf{e}_3) = \mathbf{e}_3$, it isn't moving because we're rotating around it. $T(\mathbf{e}_1) = \mathbf{e}_2$, as the x direction moves to the y direction. Finally $T(\mathbf{e}_2) = -\mathbf{e}_1$, as y moves to x, but negative. We can put this together into matrix $Z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

d. Before I do the answer, I should find the negative around x matrix. It's like X, but this time $T(\mathbf{e}_2) = -\mathbf{e}_3$ and $T(\mathbf{e}_3) = \mathbf{e}_2$. So, we have $N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$. So, to find the effect of $90°$ around x, then $90°$ around y, and finally $-90°$ around x, we compute $NYX = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which you should notice is very close to Z, but not quite. It's Z with the signs switched; this is a rotation around the z-axis but $-90°$, alright that's kinda interesting, but not what I wanted to know.

e. Ok, I was hoping that d. asked for the x then y then z result. The first person to send me that, correct, with a explanation of the geometric effect I will record $+1$ for this problem set.

§2.6.7. a. The matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is pretty simple. It multiplies all vectors by two, and hence doubles the length of all vectors, scaling them by a factor of two. In general $A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$ scales the length of the vectors by r (notice that we assume here that $r > 0$).

b. Notice by using our view of columns that If $R(\mathbf{x}) = A\mathbf{x}$ for $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ that $R\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $R\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, the x-direction goes to y and the y-direction goes to $-y$. This should all look familiar. This is our $90°$ rotation, counterclockwise (as all mathematicians hate clocks [apparently]). This is a clue for the more general $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Notice if $\theta = 90°$, then we get the first matrix. That's a big clue. Also notice that If $R(\mathbf{x}) = A\mathbf{x}$, then $R\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $R\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$. The first should look familiar from polar coordinates in calc II. The second is the $90°$ rotation of the first. I hope you can put these two facts together to see that the matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is the general rotation matrix counterclockwise thru angle θ.

c. This is simple computation:

$$\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix}$$

d. There's not much to do here. $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix}$ by substitution for a and b which equals $\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ by the work in c.

e. Connecting to polar coordinates from Calc II, remember $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(b/a)$. With these values $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is the matrix of rotation by angle θ and scaling by factor r, in either order.

f. $S \circ T$ is T first then S. This is two successive rotations and two successive scalings. They combine together to be a scaling of $\sqrt{a^2 + b^2}\sqrt{c^2 + d^2}$ and a rotation of $\tan^{-1}(b/a) + \tan^{-1}(d/c)$. That's an awkward way to say it, but it's the two rotation/scalings done one after the other.