233 Problem Set 3 Solutions

§3.1 5. This is either second or third time we've seen this idea. The first time was at the start of the previous problem set, §2.2.7. This I say is the second time, and the third time was in a recent activity, §3.4.3. If you put off this work until afterward, then you missed that connection.

a. For $A = \begin{bmatrix} 0 & 2 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ our first move is to switch rows and then multiply by scalars to get the diagonals to ones. The matrix that switches rows one and two is: $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ this makes our so far matrix $A' = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ we now scale each of the rows by $E_2 = \begin{bmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. From what we were told $A^{-1} = E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Ok, that's one.

b. We start with $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$. One by one, we add negative two times the first to the second using matrix $E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ to get $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$ and then three times the second to the third using matrix $E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ to get $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$ and finally negative two times the third to the fourth using matrix $E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}$ to get the Identity. Again $A^{-1} = E_3E_2E_1 =$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -6 & 3 & 1 & 0 \\ 12 & -6 & -2 & 1 \end{bmatrix}$. We have one more inverse.

c. One more. This time $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. Again three steps. The first column looks good. So, we start by using $E_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to add negative one times the second row to the first to get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. Next scale the last row by half using $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$ to get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Now we need to make one more zero, so we add negative one times the last row to the second using $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ to get the Identity. One last time $A^{-1} = E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$. And, we're done.

All of this can be checked in Sage by asking it to find the inverses. I hope you were responsible about doing so (I did).

§3.1 11. Now a step-by-step process. Tiny steps.

a. 1. AB is invertible. Let's say the inverse is called M. Hence $MAB = I$. We are trying to show "if $AB\mathbf{x} = \mathbf{0}$ then $\mathbf{x} = \mathbf{0}$." So, suppose $AB\mathbf{x} = \mathbf{0}$ and multiply both sides on the left by M. This gives $MAB\mathbf{x} = I\mathbf{x} = \mathbf{x}$ on the left and $M\mathbf{0} = \mathbf{0}$ on the right, hence $\mathbf{x} = \mathbf{0}$.

2. We going to use "if $AB\mathbf{x} = \mathbf{0}$ then $\mathbf{x} = \mathbf{0}$." We start with $B\mathbf{x} = \mathbf{0}$ and multiply both sides on the right by A so that we can have AB. This gives $AB\mathbf{x} = A\mathbf{0} = \mathbf{0}$. So, we started with $B\mathbf{x} = \mathbf{0}$, and now we have $AB\mathbf{x} = \mathbf{0}$ and we know that "if $AB\mathbf{x} = \mathbf{0}$ then $\mathbf{x} = \mathbf{0}$." so we use that to find $\mathbf{x} = \mathbf{0}$, as expected.

3. If $B\mathbf{x} = \mathbf{0}$ this is asking what linear combinations of the columns produce $\mathbf{0}$. The fact that the only one is $\mathbf{x} = \mathbf{0}$, tells us that the vectors are independent. Because there are n columns, and they are independent, they span $\mathbb{R}^n$. So, we have a pivot in every column, therefore B is invertible.

b. 1. AB is invertible so again there is an inverse, I'm still calling it M. If $AB\mathbf{x} = \mathbf{b}$, we can multiply both sides by M to get $MAB\mathbf{x} = I\mathbf{x} = \mathbf{x}$ on the left. On the right we get $M\mathbf{b}$ which tells us $\mathbf{x} = M\mathbf{b}$. Hence there is a solution, hence $AB\mathbf{x} = \mathbf{b}$ is consistent.

2. Suppose we have $A\mathbf{y} = \mathbf{b}$. I intentionally changed the letter, because I don't want to get confused with what is about to come. We know from 1. that $A(B\mathbf{x}) = \mathbf{b}$ is consistent. Therefore $AB\mathbf{x} = \mathbf{b}$ has a solution, so there is a $B\mathbf{x}$ that satisfies this equation. So, there is a $\mathbf{y} = B\mathbf{x}$ so that $A\mathbf{y} = A(B\mathbf{x}) = AB\mathbf{x} = \mathbf{b}$, so $A\mathbf{y} = \mathbf{b}$ is consistent.

3. All of 1. and 2. is for every vector $\mathbf{b}$. For this to be true, A must reduce to the identity with a pivot in every row so that we can have a solution for every $\mathbf{b}$. Therefore, A must be invertible.

So, altogether we see ... if AB is invertible, and both A and B are square, then A and B are also invertible.

§3.2 3. a. There's lots of ways to see if this forms a basis. There are four vectors. We need to show they are independent. One way we may do this is to make them into a matrix and row reduce. This works pretty quickly, just subtracting the first row from all the others, then the second row from the third and fourth, and finally the third from the fourth. The matrix there is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. Let's call it B. Because there are 4 vectors and they are independent, they span $\mathbb{R}^4$. An even quicker way, but using material beyond this section, is to see that because the matrix is lower triangular it's determinant is the product of the diagonal which is one not zero. This was not the intent, but it is valuable.

b. The rest here should sound familiar from activity 3.2.3. From that 3.2.3 we have $B\{\mathbf{x}\}_{\mathcal{B}} = \mathbf{x}$. So, multiply by B. Or equivalently you may use the components as weights of the original vectors to form a linear combination.

c. To undo this, we have the above equation $B\{\mathbf{x}\}_{\mathcal{B}} = \mathbf{x}$, which we solve for $\{\mathbf{x}\}_{\mathcal{B}}$ by row reducing. This is equivalent to multiplying by B^{-1}.

d. and e. were originally switched in these solutions. They are *now* correct.

d. Following c., we start with $\begin{bmatrix} 1 & 0 & 0 & 0 & 23 \\ 1 & 1 & 0 & 0 & 12 \\ 1 & 1 & 1 & 0 & 10 \\ 1 & 1 & 1 & 1 & 19 \end{bmatrix}$ and row reduce to get $\begin{bmatrix} 23 \\ -11 \\ -2 \\ 9 \end{bmatrix}$.

e. Ok, so, we do what we said in b. and compute $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -3 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \\ -3 \end{bmatrix}$.

So, we start with $\begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & -3 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix}$ and row reduce to get $\begin{bmatrix} 3 \\ -2 \\ -4 \\ -1 \end{bmatrix}$.

§3.2 7 a. Yes, if you have independent vectors they span a space of the same dimension as the number of vectors, but as they are in $\mathbb{R}^m$, they span a space that is m dimensional in $\mathbb{R}^m$, which is $\mathbb{R}^m$ itself.

b. If A is invertible, then there is a pivot in every row and column, hence the columns are independent and because there is a pivot in every row it spans $\mathbb{R}^m$. Therefore it is a basis.

c. If A is invertible then it has a pivot in every row and column, which means it row reduces to the identity. But, any step along the way of row reducing, e.g. B also must row reduce to the identity, hence have pivots in every row and every column. It also means that the columns are independent and span $\mathbb{R}^m$. Hence B is also composed of columns that maintain these properties, and hence are a basis for $\mathbb{R}^m$?

d. Yes, if you put these ten vectors into a 10×10 matrix, because they span $\mathbb{R}^{10}$, there is a pivot in every row. But because there are only 10 columns, there must also be a pivot in every column, therefore the vectors are independent and hence a basis.

§3.4 7 Notice this is the same as what we did a lot of work for in 3.1 11 above.

a. If A and B are both invertible then $\det A \neq 0$ and $\det B \neq 0$. The product of two nonzero numbers is not zero. So, $\det A \det B = \det(AB) \neq 0$. Because $\det(AB) \neq 0$ we then know that AB is invertible.

b. If AB is invertible, then $\det(AB) \neq 0$. But then $\det A \det B \neq 0$. We have an algebraic property: if $ab \neq 0$, then $a \neq 0$ and $b \neq 0$. Applying this tells us that $\det A \neq 0$ and $\det B \neq 0$ hence A and B are invertible.

§3.4 11. A can be reduced the identity by switching rows, it takes 3 exchanges as we move the bottom row up one by one. So, we multiply by $(-1)^3$ and then find the determinant of the identity which is 1, therefore $\det A = -1$.

How many switches do we need here to move B to the identity? We exchange first and second and also third and fourth. That is two exchanges, so we multiply by $(-1)^2$ to get to the identity, which is 1, therefore $\det B = 1$.

For C we switch the first and last rows and the second and third rows, this gives us a diagonal matrix. The determinant of a diagonal matrix can be found by multiplying the diagonal entires. Putting this all together gives $(-1)^2 abcd = abcd$, which gives out determinant.

§3.5 5. Here's the guiding principle to remember: the dimension of the null space is the number of free variables, which is the number of columns without pivots. I expect you will all give examples in reduced form, which isn't wrong, but misses some of the point. I will give both kinds of examples.

a. If dim Nul$(A) = 0$, then there are zero free variables, and three pivots. Here's the reduced echelon form example. If you insist on reducing this is the only possible example: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. In general this is any matrix with three independent columns, any matrix that is invertible, any matrix that has nonzero determinant. This can be accomplished by not being careful. Here's one I'm making up by using numbers that are highly unlikely to be related: $\begin{bmatrix} 3.14 & -2 & 7 \\ -5 & 2.72 & 13 \\ 5.8 & 16 & 9 \end{bmatrix}$. The determinant is -1487.4848, that's quite not zero.

b. For dim Nul$(A) = 1$, we need one free variable and two pivots. In reduced echelon form this is something like: $\begin{bmatrix} 1 & -4 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{bmatrix}$. I imagine the second is more common. They are very different in bases for the column space. Here are two unreduced matrices that have these reduced forms: $\begin{bmatrix} 1 & -3 & 7 \\ -2 & 6 & 9 \\ 5 & -15 & 12 \end{bmatrix}$, the in the second form is the one that people often arbitrarily make as an example: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ (it's easy to type into sage). What do I mean by "They are very different in bases for the column space."? I will give +1 on this problem set for anyone who sends me a well-written explanation for this meaning before our upcoming quiz. I guess there's also the option of the third column being all zeroes. Include that option in your analysis also, and explain how it is different from the other two.

c. For dim Nul(A) = 2, we need two free variables and one pivot. This is starting to have fewer options. The reduced eschelon form looks like $\begin{bmatrix} 1 & 3 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The unreduced examples need to have three columns that are multiples of each other. There's not much use in being terribly creative there. Here's one: $\begin{bmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ -4 & -12 & 8 \end{bmatrix}$. There could be one or even two columns of zeroes.

d. For dim Nul(A) = 3, we need three free variables and no pivots. Now we're out of options. There is only one possible 3×3 matrix that satisfies this: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, that's the only way to get zero pivots.

§3.5 9. Ok, we start with $A = P \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} P^{-1}$.

a. We want to know about Nul(A). This is the set of all vectors $\mathbf{v}$ such that $A\mathbf{v} = \mathbf{0}$. We may now think about solving for $\mathbf{v}$. We know $P \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} P^{-1}\mathbf{v} = \mathbf{0}$. And we solve by undoing from the left, so let's begin by multiplying both sides by P^{-1} on the left. This gives

$$P^{-1}P \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} P^{-1}\mathbf{v} = I \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} P^{-1}\mathbf{v} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} P^{-1}\mathbf{v} = P^{-1}\mathbf{0} = \mathbf{0}$$

From this we take away $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} P^{-1}\mathbf{v} = \mathbf{0}$.

Next we notice that $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is invertible, either because it's determinant is -6, because it has three pivots, or because its inverse is $\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so we can multiply both sides by its inverse on the left also. This should be looking predictable by now. Doing so we get:

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} P^{-1}\mathbf{v} = IP^{-1}\mathbf{v} = P^{-1}\mathbf{v} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{0} = \mathbf{0}$$

taking away $P^{-1}\mathbf{v} = \mathbf{0}$.

Finally we can multiply both sides on the left by P to get $PP^{-1}\mathbf{v} = I\mathbf{v} = \mathbf{v} = \mathbf{0}$, which tells us the null space is only the zero vector. Nul(A) = $\{\mathbf{0}\}$. That answers part a.

b. Ok, let's connect with what we saw in 5. We just found that the dimension of the null space is 0, so there are no free variables, so there are 3 columns with pivots, so the column vectors are independent, hence the column space, Col(A) is all of $\mathbb{R}^3$.

For the upcoming quiz, and the soon following exam, please fight to *understand* this material. Words here matter.