233 Problem Set 4 Solutions

§4.1 6. a. This is not *so* bad, but .. surprising that it's not more scaffolded. Step one, write $\mathbf{x}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, this probably is worth doing by inspection instead of row reduction $\begin{bmatrix}5\\0\end{bmatrix} = 2\begin{bmatrix}2\\1\end{bmatrix} - 1\begin{bmatrix}-1\\2\end{bmatrix}$. So, $A\mathbf{x} = A(2\mathbf{v}_1 - 1\mathbf{v}_2) = 2A\mathbf{v}_1 - A\mathbf{v}_2 = 2(2)\mathbf{v}_1 - 1(-3)\mathbf{v}_2 = 4\mathbf{v}_1 + 3\mathbf{v}_2$. I could compute this out, but, remember, this wasn't the actual question. The actual question is for A^4 instead. This is where we can see some of the power of eigenvalues. $A^4\mathbf{x} = A^4(2\mathbf{v}_1 - 1\mathbf{v}_2) = 2A^4\mathbf{v}_1 - A^4\mathbf{v}_2 = 2(2^4)\mathbf{v}_1 - 1(-3)^4\mathbf{v}_2 = 32\mathbf{v}_1 - 81\mathbf{v}_2 = 32\begin{bmatrix}2\\1\end{bmatrix} - 81\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}145\\-130\end{bmatrix}$.

b. $\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{5}\begin{bmatrix}5\\0\end{bmatrix} = \frac{2}{5}\begin{bmatrix}2\\1\end{bmatrix} - \frac{1}{5}\begin{bmatrix}-1\\2\end{bmatrix}$, so $A\begin{bmatrix}1\\0\end{bmatrix} = A\left(\frac{2}{5}\begin{bmatrix}2\\1\end{bmatrix} - \frac{1}{5}\begin{bmatrix}-1\\2\end{bmatrix}\right) = \frac{4}{5}\begin{bmatrix}2\\1\end{bmatrix} + \frac{3}{5}\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$. That was not that bad. Now, what about the other one? I want something nice that is a multiple of $\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{5}\begin{bmatrix}0\\5\end{bmatrix} = \frac{1}{5}\left(\begin{bmatrix}2\\1\end{bmatrix} + 2\begin{bmatrix}-1\\2\end{bmatrix}\right) = \frac{1}{5}\begin{bmatrix}2\\1\end{bmatrix} + \frac{2}{5}\begin{bmatrix}-1\\2\end{bmatrix}$ So, $A\begin{bmatrix}0\\1\end{bmatrix} = A\left(\frac{1}{5}\begin{bmatrix}2\\1\end{bmatrix} + \frac{2}{5}\begin{bmatrix}-1\\2\end{bmatrix}\right) = \left(\frac{2}{5}\begin{bmatrix}2\\1\end{bmatrix} - \frac{6}{5}\begin{bmatrix}-1\\2\end{bmatrix}\right) = \begin{bmatrix}2\\-2\end{bmatrix}$

c. The results in b. give us the columns of $A = \begin{bmatrix}1 & 2\\2 & -2\end{bmatrix}$. We can check this by multiplying A by $\mathbf{v}_1$ and $\mathbf{v}_2$.

§4.1 9 This is intended to be done visually and geometrically. I will also include the matrices if you wish to confirm that way. We are looking for vectors that point in the same (or opposite) direction after the transformation.

a. The line $y = x$ doesn't move in this reflection. Any vector on it, e.g. $\begin{bmatrix}1\\1\end{bmatrix}$ is an eigenvector, since it doesn't move. The eigenvalue associated with this vector is 1, since it doesn't change, i.e. multiplies by 1. Any multiple of this vector is also an eigenvector with an eigenvalue of 1. Are there vectors that are sent to their opposites? Yes. The vector $\begin{bmatrix}1\\-1\end{bmatrix}$ reflects to $\begin{bmatrix}-1\\1\end{bmatrix}$. Because they are sent to their opposites, they multiply by -1, which is the eigenvalue. Again any multiple is an eigenvector with eigenvalue -1. I forgot, but now including the matrix, as I said I would. The matrix here is $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$.

b. For a half-turn rotation in the plane, no (nonzero) vectors have the same positive direction before and after, but there are plenty that go to their negative. They include $\begin{bmatrix}1\\0\end{bmatrix}$ which goes to $\begin{bmatrix}-1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ which goes to $\begin{bmatrix}0\\-1\end{bmatrix}$. But, in fact, this is true for *all* vectors. Every vector: $\begin{bmatrix}x\\y\end{bmatrix}$ goes to $\begin{bmatrix}-x\\-y\end{bmatrix}$, so *every* vector is an eigenvector with -1 eigenvalue. That's pretty extreme. To be clear, what we're looking for in "describe" is a basis, so as long as you gave me *two* vectors that are independent (i.e. not multiples), that will be great. The matrix here is $-I = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$.

c. This is another half-turn, which will have some of the same features. This a half-turn rotation around the y-axis in $\mathbb{R}^3$. The y-axis doesn't move, so the vector $\begin{bmatrix}0\\1\\0\end{bmatrix}$ goes to itself. This is an eigenvector with eigenvalue 1, as are all multiples. For the rest, we have a situation like above in b., The x-direction goes to its negative, and so does the z-direction, but also any combination of them. To be complete, any vector $\begin{bmatrix}x\\0\\z\end{bmatrix}$ goes to $\begin{bmatrix}-x\\0\\-z\end{bmatrix}$, but shorter and better to give a basis, you need two vectors that are independent (not multiples) but which have zero in the y-coordinate. The obvious ones are $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\end{bmatrix}$. These all have eigenvalue -1. The matrix here is $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

d. We're rotating a quarter-turn around the x-axis. So, the x direction doesn't change (like in c.) hence one eigenvector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and all multiples. This has eigenvalue 1. The catch is in this case, like in activity 4.1.2.h. where we did this in the plane, if we rotate a quarter turn, no nonzero vectors are in the same or opposite direction. Therefore all vectors with nonzero coordinates in y and z change, and there are no more eigenvectors here. The matrix here is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$. Oh, this is fun ... so, we have seen (in 4.1.2.h) a geometric example in $\mathbb{R}^2$ that has *no eigenvectors*. This example here in $\mathbb{R}^3$ has only one direction of eigenvectors, but it has that one direction. Ok, here's the extra this time, and it's different. The goal: find a geometric transformation in $\mathbb{R}^3$ that has no eigenvectors. Wait a minute - this is an extra level of interesting. This is *not possible*. How about this, I will not have you write exam solutions (no matter how it goes - so - be prepared). But, I will accept an explanation for why this is not possible, submitted in groups (no larger than three people with attestation that the people all contributed), no later than class-time on 7 November, a week after the exam. I will give one point on XM2 for a serious attempt and two if it is correct and well-written.

§4.2 5. This is pretty exclusively sage-work. The point here is to be exposed to what is possible.

a. Sage says "`[(3, [(0, 1, 2)], 1), (2, [(1, 0, -1)], 1), (-1, [(1, -1/2, -3/2)], 1)]`". Writing that is not enough, interpreting it is important. There are three eigenvalues, 3, 2, and -1. They each have one dimension associated eigenvector and each have multiplicity one. The eigenvectors are $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$ (why did Sage use fractions?) respectively. There are three of them, so they are a basis for all $\mathbb{R}^3$.

b. Sage says "`[(1, [(1, -1/2, -3/2)], 1), (-1, [(1, 1, 1)], 2)]`". This says there are two eigenvalues, 1 and -1. The first has multiplicity one but the second has multiplicity two. Multiplicity two says it *could have* two dimensions of eigenvectors, but .. it doesn't, otherwise sage would have given us two vectors as a basis for the eigenvectors for -1. So, the eigenvector for 1 is $\begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$ (and multiples) [that's an odd coincidence from a.], and for -1 is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (and multiples). We do *not* have a basis for $\mathbb{R}^3$ here.

c. Sage says "`[(1, [(1, -1/2, -3/2)], 1), (-1, [(1, 0, -1), (0, 1, 2)], 2)]`". This tells us there are two eigenvalues, ± 1. The positive eigenvalue has a one dimensional eigenspace, with basis that same vector again (hmmm): $\begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$, and the -1 eigenvalue has a two dimensional eigenspace with basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$. That's three vectors so there is a basis of eigenvectors for $\mathbb{R}^3$ in this case.

§4.2 8. This should look familiar, both from an activity and a problem set. This is a primary reason that eigenvectors are of practical use. Since there isn't a sage box here, I'd like if you show work. This is a chance to try out our magical equation, $\det(A - \lambda I) = 0$. In this case, to start by finding the determinant of $\begin{bmatrix} 0.4-\lambda & 0.3 \\ 0.6 & 0.7-\lambda \end{bmatrix}$. The determinant is $(0.4-\lambda)(0.7-\lambda) - 0.3(0.6) = \lambda^2 - 1.1\lambda + 0.1 = (\lambda - 1)(\lambda - 0.1)$. So our two eigenvalues are 1 and 0.1. Now to find the eigenvectors. Compute $A - I$ and $A - 0.1I$ to get:

$\begin{bmatrix} -0.6 & 0.3 \\ 0.6 & -0.3 \end{bmatrix}$ and $\begin{bmatrix} 0.3 & 0.3 \\ 0.6 & 0.6 \end{bmatrix}$. The first one produces $x = y/2$, and the second one $x = -y$ which gives eigenvectors of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, respectively. There's only one dimension, so those are a basis.

b. $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{3}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$.

c. This reminds me of the first problem. I'm not sure why it was there, but I am sure why it's here. In any case, we do the same thing: to follow along, let's call $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So, $\mathbf{x}_1 = A\mathbf{x}_0 = A(\frac{1}{3}\mathbf{v}_1 - \frac{1}{3}\mathbf{v}_2) = \frac{1}{3}A\mathbf{v}_1 - \frac{1}{3}A\mathbf{v}_2 = \frac{1}{3}\mathbf{v}_1 - \frac{1}{30}\mathbf{v}_2$. Similarly, each time we apply A we multiply by the eigenvalues, giving $\mathbf{x}_2 = A^2\mathbf{x}_0 = \frac{1}{3}\mathbf{v}_1 - \frac{1}{300}\mathbf{v}_2$, and lastly $\mathbf{x}_3 = A^3\mathbf{x}_0 = \frac{1}{3}\mathbf{v}_1 - \frac{1}{3000}\mathbf{v}_2$

d. In general $\mathbf{x}_k = A^k\mathbf{x}_0 = \frac{1}{3}\mathbf{v}_1 - \frac{1}{3*10^k}\mathbf{v}_2$. So, as k increases the last term is going to zero and the limiting case is approaching $\frac{1}{3}\mathbf{v}_1$, i.e. $\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$. We aren't told this in context, but this must be a probability situation, like our cars and bicycles. This says in the long run $\frac{1}{3}$ will end up in the first state and $\frac{2}{3}$ in the second.

§4.3 6. Similarity is a topic that has appeared in the exercises a bit, but this is our first time with it. A is similar to B if there is an invertible (must be from the equation) matrix P such that $A = PBP^{-1}$.

a. If A is similar to B then $A = PBP^{-1}$, but we can solve this for B by multiplying by P and P^{-1}, thus we see that $B = P^{-1}AP = P^{-1}AP^{-1^{-1}}$ and so P^{-1} plays the role of P.

b. If A is similar to B then $A = PBP^{-1}$, If B is similar to C then $B = QCQ^{-1}$. Substituting for B in the expression for A we get $A = PQCQ^{-1}P^{-1}$, so A is similar to C with PQ playing the role of P, remember $(PQ)^{-1} = Q^{-1}P^{-1}$.

c. To recall, B is diagonalisable iff $B = RDR^{-1}$. If A is similar to B then $A = PBP^{-1}$, substituting for B again we get $A = PRDR^{-1}P^{-1}$ so, A is diagonalisable this time using PR.

d. This is more interesting. We consider $A = PBP^{-1}$, but now extend it to $B - \lambda I$, and see $P(B - \lambda I)P^{-1} = PBP^{-1} - P\lambda IP^{-1} = A - \lambda PP^{-1} = A - \lambda I$. This is the key. Because of this We can start with $\det(P(B - \lambda I)P^{-1}) = \det(P)\det(B - \lambda I)\det(P^{-1}) = \det(B - \lambda I)\det(P)\det(P^{-1}) = \det(B - \lambda I)\det(PP^{-1}) = \det(B - \lambda I)\det(I) = \det(B - \lambda I)1 = \det(B - \lambda I)$. That's half of what we need (and a lot of extra steps, it's ok if you skip some of them), but from above we see that $\det(P(B - \lambda I)P^{-1}) = \det(A - \lambda I)$, so $\det(A - \lambda I) = \det(B - \lambda I)$ as wanted.

e. After the last one, which was the work for this question, this one is easy. If they have the same characteristic polynomial, then they have the same roots, which are the eigenvalues.

§4.3 10. Notice these matrices are all diagonal, so the multiplication is easy, just multiply the diagonal by the corresponding entry.

a. I'm going to call the vectors $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{x}_1 = \begin{bmatrix} 1.4 \\ 1.4 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1.96 \\ 0.98 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 2.744 \\ 0.686 \end{bmatrix}$, and $\mathbf{x}_4 = \begin{bmatrix} 3.8416 \\ 0.4802 \end{bmatrix}$. I will try to draw pictures and include them. I put them at the end. They shouldn't be surprising.

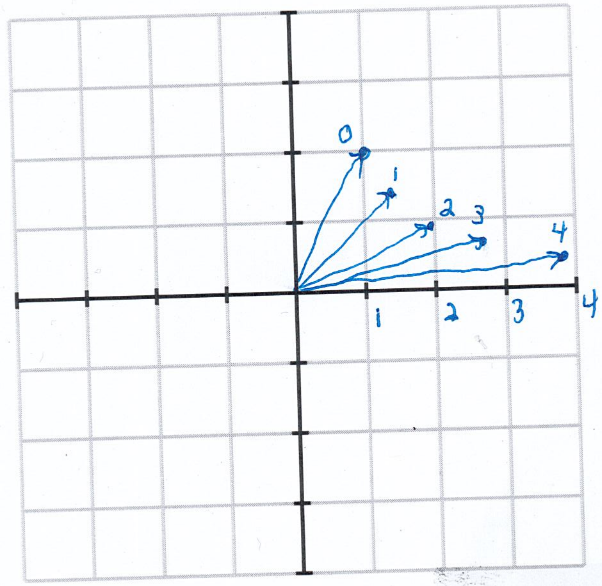
b. Same computations: $\mathbf{x}_0 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, $\mathbf{x}_1 = \begin{bmatrix} 2.4 \\ 2.7 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1.44 \\ 2.43 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 0.864 \\ 2.187 \end{bmatrix}$, and $\mathbf{x}_4 = \begin{bmatrix} 0.5184 \\ 1.9683 \end{bmatrix}$.

c. Again $\mathbf{x}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{x}_1 = \begin{bmatrix} 2.4 \\ 1.4 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 2.88 \\ 1.96 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 3.456 \\ 2.744 \end{bmatrix}$, and $\mathbf{x}_4 = \begin{bmatrix} 4.1472 \\ 3.8416 \end{bmatrix}$.

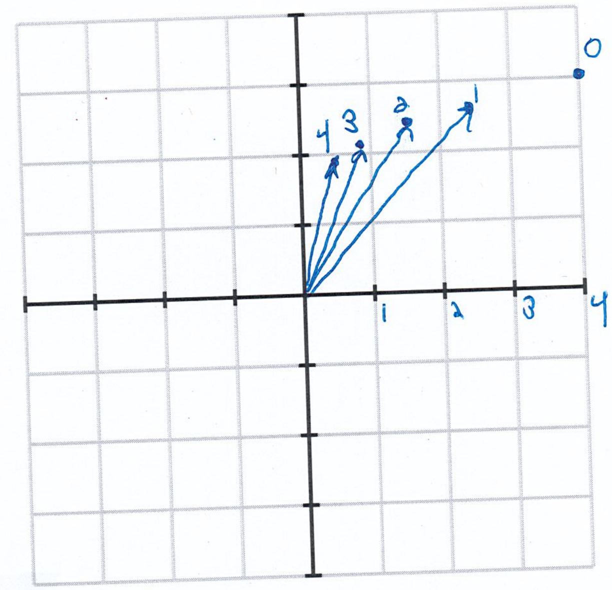
d. Ok, we said we could use sage for anything above. I don't see the point in struggling to find eigenvalues and eigenvectors now, so ... we have an eigenvalue of 1.2 with eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and an eigenvalue of 0.7 with eigenvector $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. I wish that the vector $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ that would be easy, but alas, it's $\mathbf{x} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 1.5\mathbf{v}_1 + 1.5\mathbf{v}_2$. By now we're used to this drill, $\mathbf{x}_k = A^k = 1.5(1.2^k)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1.5(0.7)^k\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So, $\mathbf{x}_0 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\mathbf{x}_1 = \begin{bmatrix} 2.85 \\ 0.75 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 2.895 \\ 1.425 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 3.1065 \\ 2.0775 \end{bmatrix}$, and $\mathbf{x}_4 = \begin{bmatrix} 3.47055 \\ 2.75025 \end{bmatrix}$.

e. This is the easiest part. We see this pattern, or similar, in b. As k gets larger the vectors get smaller converging to $\mathbf{0}$. That's the whole answer. In c. they will get arbitrarily large. In a. they will get closer to the positive x-axis. In d. they get arbitrarily large, but the coordinates converge together, as we're getting closer to a multiple of the first eigenvector.

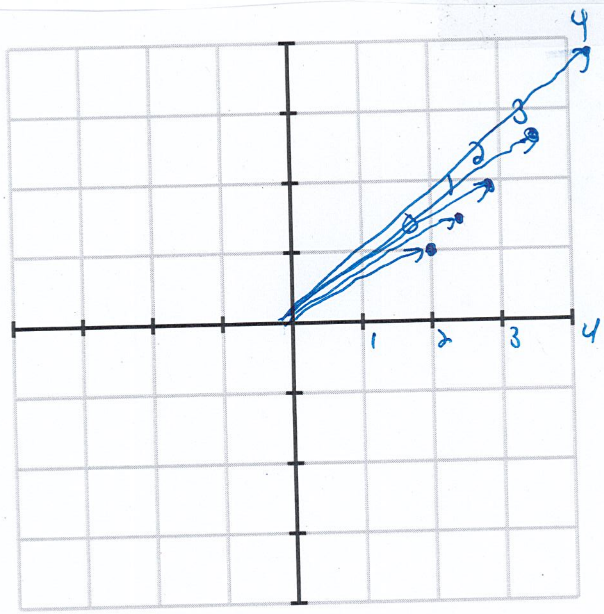
a.



b.



c.



d.