233 Problem Set 6 Solutions

6.1.5. Let's work left to right. This is probably just going to be a long string of equations. $|\mathbf{v}+\mathbf{w}|^2 + |\mathbf{v}-\mathbf{w}|^2 = (\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}+\mathbf{w}) + (\mathbf{v}-\mathbf{w})\cdot(\mathbf{v}-\mathbf{w}) = (\mathbf{v}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}+\mathbf{w}\cdot\mathbf{w}) + (\mathbf{v}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{w}-\mathbf{v}\cdot\mathbf{w}+\mathbf{w}\cdot\mathbf{w})$ lots cancels $= 2\mathbf{v}\cdot\mathbf{v} + 2\mathbf{w}\cdot\mathbf{w} = 2|\mathbf{v}|^2 + 2|\mathbf{w}|^2$, and that's it, this is surprisingly not bad.

For the picture it's almost as if the sum of the diagonals is equal to the perimeter of the parallelogram, but ... not quite. There's squares around everything. So, this says that the sum of the squares of the diagonals is equal to the sum of the squares of the sides of the parallelogram.

6.1.6a. $\mathbf{x}\cdot\mathbf{v}_1 = 0$ because they are perpendicular. In particular this is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} = 2x + 0y + 4z = 0.$ There's our equation.

b. I expect that I don't need to show you how we get the second equation, which is $-1x + 2y - 4z = 0$, so those two equations are our linear system.

c. Ok, so now we have some work, let's put this into a matrix: $\begin{bmatrix} 2 & 0 & 4 & 0 \\ -1 & 2 & -4 & 0 \end{bmatrix}$. This reduces to $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$ which gives us $x = -z$, $y = z$ and $z = z$ i.e. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} z$. This represents a line, the line that is perpendicular to the plane spanned by bfv_1 and $\mathbf{v}_2$. I don't expect much from the picture more than it being a line perpendicular to a plane.

d. Ok, we do the same thing, but this time just to the first equation. We just divide by two to get $x + 2z = 0$, so we have $x = -2z$, $y = y$, and $z = z$, so we have $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} z$. We now have the orthogonal complement is a plane, perpendicular to the line of the original vector.

6.2.5 (originally assigned and retracted) a. $A^T(BA^T)^{-1} = A^T(A^T)^{-1}B^{-1} = IB^{-1} = B^{-1}$.

b. $(A+B)^T(A+B) = (A^T+B^T)(A+B) = A^TA + A^TB + B^TA + B^TB$. Is that simpler? I think the original was simpler. That one is odd.

c. $[A(A+B)^T]^T = [A(A^T+B^T)]^T = [AA^T + AB^T]^T = (AA^T)^T + (AB^T)^T = A^{T^T}A^T + B^{T^T}A^T = AA^T + BA^T$, again, fine, I guess. I'm still not seeing the point.

d. $(A+2I)^T = A^T + 2I^T = A^T + 2I$ ok, yes, it's valuable that identity doesn't change, but ... so?

6.2.6. a. When we take the transpose of a matrix, we interchange the rows and columns. This interchanges the dimensions. If we are to get the same exact matrix then at the very least, the shape must stay the same, which means the matrix must have the same number of rows and columns, i.e. must be a square.

b. 1. If you take the transpose of a diagonal matrix, D^T, because the diagonal doesn't move, and because the other entries are all zero, you don't notice taking the transpose and $D^T = D$.

2. $[BAB^{-1}]^T = (B^{-1})^TA^TB^T$. That looks pretty different, so probably these are not always the same. To show that it's not guaranteed, let's make the simplest example we can (to keep our work easy). First, B isn't really the issue, so to keep that easy, let $B = I$, the identity matrix, which is its own inverse. Notice that A changes to the transpose, A^T. So, all we need is a matrix A that isn't symmetric. My choice is $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. So, we put this all together, $BAB^{-1} = IAI = A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. On the other hand, $[BAB^{-1}]^T = (B^{-1})^TA^TB^T = IA^TI = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, and they are not equal. Of note here, we kinda ignored $(B^{-1})^T$, but there's no general reason that that is particularly nice. We chose a very simple matrix for B so we didn't need to worry about it, but unless B is orthogonal (where $B^{-1} = B^T$), this matrix $(B^{-1})^T$ is just some complicated thing you get by first taking the inverse and then the transpose.

3. $(AA^T)^T = (A^T)^TA^T = AA^T$, these are equal.

4. $(BDB^T)^T = (B^T)^TD^TB^T = BDB^T$, notice in the last step we used part 1., that $D^T = D$. So, these are equal also.

6.2.7. The key to this problem is the fact (that we discussed in class during activities), that $\det(M) = \det(M^T)$. It is a powerful and important fact that gives us several new ways to compute determinants.

a. We're looking at the characteristic polynomial of A^T, $\det(A^T - \lambda I)$, but before that let's look at just $A - \lambda I$. Notice $(A - \lambda I)^T = A^T - \lambda I^T = A^T - \lambda I$. So, let's use this along with $\det(M) = \det(M^T)$. Here are some steps:

$$\det(A^T - \lambda I) = \det((A - \lambda I)^T) = \det(A - \lambda I)$$

The first is the characteristic polynomial of A, and the last is the characteristic polynomial of A^T. The first equal is by the "Notice" statement, and the second is by the transpose doesn't affect determinant fact.

Here's a different direction to the same end that you may like more, using the transpose determinant fact first instead:

$$\det(A^T - \lambda I) = \det((A^T - \lambda I)^T) = \det(A - \lambda I^T) = \det(A - \lambda I)$$

I am hopeful that since there are several paths here that you will find one you like.

b. If they have the same polynomial, they have the same roots. There's nothing more here.

c. $A = PDP^{-1}$ is given. So, we can find a diagonalisation by taking transpose of the whole thing, $A^T = (P^{-1})^T D^T P$, and that would be good enough, but remember that $D^T = D$ so this equals $(P^{-1})^T D^T P$. Remember that the first matrix in the diagonalisation has columns of eigenvectors. This is interesting because it says if you have a matrix with columns of eigenvectors of A, and take its inverse transpose, you get a matrix of eigenvectors of A^T. A somewhat curious connection indeed.

6.3.7 a. $QQ^T\mathbf{w}$ is the projection of vector $\mathbf{w}$ onto the subspace W. The projection of a vector *in* the subspace is the vector itself. Notice that $\mathbf{w} = \hat{\mathbf{w}} + \mathbf{0}$, and $\mathbf{0}$ is definitely perpendicular to $\mathbf{w}$ since it is perpendicular to everything. This is true, $QQ^T\mathbf{w} = \mathbf{w}$.

b. This is true. Orthogonal vectors are independent. If there were more than 8 independent vectors in $\mathbb{R}^8$, then $\dim(\mathbb{R}^8) > 8$, but $\dim(\mathbb{R}^8) = 8$.

c. This is false. Here's the simplest counterexample: (and a lot of typing, but ok) $Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

From there $Q^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$. Now putting them together, in this way we have

$$QQ^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which is not the identity, it has zeroes in the right corner.

d. This is true, we saw it in activity 6.3.4.b.6. Here is a way to see it. $Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix}$. So $Q^T = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_n \end{bmatrix}$. If we multiply Q^TQ each of the entries is a dot product $\mathbf{u}_i \cdot \mathbf{u}_j$ which is 1 on the diagonal where $i = j$ because the vectors have unit length and 0 off the diagonal where $i \neq j$ because the vectors are perpendicular.

e. $\mathbf{b}$ is in $W^\perp$ if $\mathbf{b} \perp \mathbf{w}$ for all $\mathbf{w} \in W$. Remember that $\hat{\mathbf{b}}$ is computed by a sum with numerators $\mathbf{b} \cdot \mathbf{w}_i$. The only way the sum can be zero is if they are all zero. That says that $\mathbf{b}$ is perpendicular to all $\mathbf{w}_i$, hence it is also perpendicular to any linear combination, which gives all of W. I think this question is the trickiest one to say more than " it's true because it is". I look forward to seeing your answers.

6.3.8 This question is not so bad, actually. It's kinda relief. That's because they broke it down step-by-step for us.

a. We will use $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T\mathbf{w}$. Starting from the left $Q\mathbf{x} \cdot Q\mathbf{y} = (Q\mathbf{x})^T(Q\mathbf{y}) = \mathbf{x}^T Q^T Q\mathbf{y} = \mathbf{x}^T I\mathbf{y} = \mathbf{x}^T\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$. In the middle we used the associative property and the fact that we saw in the prior question that $Q^TQ = I$ for orthogonal matrices.

b. Uses a. Remember that $|\mathbf{x}|^2 = \mathbf{x}\cdot\mathbf{x}$. I will use the squares, with the step of taking square roots coming last. $|Q\mathbf{x}|^2 = Q\mathbf{x} \cdot Q\mathbf{x} = \mathbf{x} \cdot \mathbf{x} = |\mathbf{x}|^2$. The middle step is from a. Remember magnitudes are non-negative, so taking square roots isn't a problem.

c. This comes from the definition of eigenvalue, $Q\mathbf{v} = \lambda\mathbf{v}$ for a non-zero $\mathbf{v}$. Now use b., take magnitudes: $|Q\mathbf{v}| = |\lambda\mathbf{v}|$, but from b, $|Q\mathbf{v}| = |\mathbf{v}|$, so $|\lambda\mathbf{v}| = |\mathbf{v}|$, but $|\lambda\mathbf{v}| = |\lambda||\mathbf{v}|$, hence $|\lambda||\mathbf{v}| = |\mathbf{v}|$, since $\mathbf{v}$ is not zero, we can divide it from both sides to get $|\lambda| = 1$, this is only satisfied with $\lambda = \pm 1$ as desired. Probably this work is more longwinded than needed for this one.

6.4.1. Notice that the problem numbers for 6.4 are low. This is intentional. These problems are mostly straightforward application of the method in activity 6.4.2 (which is again stated in 6.4.4). Here we go.

a. $\mathbf{v}_1 = \mathbf{w}_1$ is orthogonal as one vector. The problem is $\mathbf{v}_2$. We seek the part of $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$ that is perpendicular to $\mathbf{v}_1$, so we project $\mathbf{v}_2$ onto $\mathbf{v}_1$ and subtract that. The projection is $\frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \frac{-3}{3}\mathbf{v}_1 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$. This is the part in the direction of $\mathbf{v}_1$. We subtract that, hence add one to each to find $\mathbf{w}_2 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$. Notice that the vectors are now perpendicular.

b. Divide by their lengths. Neither is pretty, we have $\sqrt{3}$ and $\sqrt{6}$.

c. $Q = \begin{bmatrix} 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix}$. I don't feel like typing the transpose. I trust you can handle that step. Multiplying by the transpose gives $QQ^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$.

d. At least the last step is short: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

6.4.4. Ok, I did the last one all by hand. There's a Sage box here. I'll try doing this more by Sage. It's the same, just not particularly interesting.

a. $\mathbf{w}_1 = \mathbf{v}_1$, $\mathbf{w}_2 =$ `v2-projection(v2,[v1])` $= \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}$

$\mathbf{w}_3 =$ `v3-projection(v3,[v1,w2])` $= \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix}$

$\mathbf{u}_1 =$`unit(v1)`$= \begin{bmatrix} 1/\sqrt{5} \\ -1/\sqrt{5} \\ -1/\sqrt{5} \\ 1/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$, $\mathbf{u}_2 =$`unit(w2)`$= \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$, $\mathbf{u}_3 =$`unit(w3)`$= \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$

`QT=matrix([u1,u2,u3])` $= \begin{bmatrix} 1/\sqrt{5} & -1/\sqrt{5} & -1/\sqrt{5} & 1/\sqrt{5} & 1/\sqrt{5} \\ 1/2 & 0 & 1/2 & -1/2 & 1/2 \\ 1/2 & 0 & 1/2 & 1/2 & -1/2 \end{bmatrix}$

$$P = \begin{bmatrix} 7/10 & -1/5 & 3/10 & 1/5 & 1/5 \\ -1/5 & 1/5 & 1/5 & -1/5 & -1/5 \\ 3/10 & 1/5 & 7/10 & -1/5 & -1/5 \\ 1/5 & -1/5 & -1/5 & 7/10 & -3/10 \\ 1/5 & -1/5 & -1/5 & -3/10 & 7/10 \end{bmatrix}$$

$$\hat{\mathbf{b}} = P\mathbf{b} = \begin{bmatrix} -9 \\ -1 \\ -11 \\ 7 \\ -5 \end{bmatrix}$$

And finally use `rref` to find $\hat{\mathbf{b}} = 2\mathbf{v}_1 - 3\mathbf{v}_2 - 1\mathbf{v}_3$. Wow, that's ... a lot of Sage. I'm glad to have it, and a bit glad to be done with it. I'm also comforted that this all worked and I got nice answers.

I notice along the way and many of you notice also ... there's a subtle difference in something with Sage. If we ask for `projection(v2,[v1])` we get $\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$, which looks to be the right answer for projecting $\mathbf{v}_2$ onto $\mathbf{v}_1$. Fine, but if we leave off those brackets around $\mathbf{v}_1$ like this: `projection(v2,v1)` we get $\begin{bmatrix} 10 \\ 5 \\ 20 \\ -20 \\ 10 \end{bmatrix}$. For four extra points on this problem set, be the first person to send me an email telling me correctly what the second computation is computing. It's clearly *not* the first answer, which I believe is the correct answer. I am asking this because I honestly don't know. I'm not putting a deadline on this one, because I sincerely want someone to figure this out and tell me.