233 Problem Set VS Solutions

Here's the catch, when I give you choice, it means more work for me. Oh, well. You deserve it. For each of these first four, look at the list of properties and see if you can find a problem.

1. First one $(a,b)+(c,d)=(a+d,b+c)$. Does this commute? I'm not sure. Let's try $(1,2)+(3,5)=(6,5)$ and $(3,5)+(1,2)=(5,6)$. Ok, one is done. If you picked the first one this was pretty quick. Two points to notice: 1. it works for $(1,2)+(3,4)$, but only because they happen to be equal. 2. $(a,b)+(c,d)=(a+d,b+c)$ and $(c,d)+(a,b)=(b+c,a+d)$ is not sufficient alone. They *look* different, but until we have actual numbers, just looking different is not enough.

2. Second one $(a,b)+(c,d)=(0,b+d)$. This is plenty commutative. I think we'll have a problem with the identity property, though. If there is an identity, call it (i,e), then $(i,e)+(a,b)=(a,b)$, but $(i,e)+(a,b)=(0,e+b)$, so all we need is a particular example where $a\neq 0$. So, I'll use $(a,b)=(1,0)$, and notice we would need: $(i,e)+(1,0)=(1,0)$ and $(i,e)+(1,0)=(0,e)$. This is not possible since $1\neq 0$.

3. This one is interesting to me. Sometimes in the list of properties needed is $(a+b)\mathbf{v}=a\mathbf{v}+b\mathbf{v}$. If this is needed then $(1+(-1))\mathbf{v}=\mathbf{v}+-1\mathbf{v}$, which is not true here, see: $(1+(-1)(1,2)=0(1,2)=(1,0)$, but $1(1,2)+-1(1,2)=(1,2)+(1,-2)=(2,0)$. But, we don't have this property listed. Sure, I'll let someone out there figure this out. Can someone prove using our shorter list of properties that this is not a vector space? +2 for the first person who does. Deadline - when I record course grades. Catch: my personal impression is that this property does *not* follow from our properties, and hence this would be considered a vector space by us (or David Austin), but not most people.

4. This one is the easiest. It is clearly not commutative: $(1,2)+(3,4)=(-2,-2)$, but $(3,4)+(1,2)=(2,2)$.

5. We saw this example in class, it was Example 1.1.7. Although I do note that the equal signs are missing there. I'm more interested in 6. Ask me if you have questions about 5.

6. We've seen vectors in $\mathbb{R}^4$, and we're used to it. It turns out infinite sequences, of the kind you studied in calc II can be thought of as $\mathbb{R}^\infty$ and make a very nice vector space. Let's think about our properties. (Despite the fact that we may actually need more than those listed, I expect none of us are checking more than those listed.). I will refer to an element of $\mathbb{R}^\infty$ as (a_n). As indicated in the problem, $(a_n)+(b_n)=(a_n+b_n)$ and $s(a_n)=(sa_n)$. We have a set with two operations and the operations are closed.

- It's commutative: $(a_n)+(b_n)=(a_n+b_n)=(b_n+a_n)=(b_n)+(a_n)$.
- The additive identity is the constant zero sequence, $(0_n)+(a_n)=(0+a_n)=(a_n)$.
- $-1(a_n)=(-a_n)$ is the additive inverse of (a_n) because $(a_n)+(-a_n)=(a_n+-a_n)=(0_n)$.
- It's also associative (why is this one this low on the list?) $((a_n)+(b_n))+(c_n)=(a_n+b_n)+(c_n)=((a_n+b_n)+c_n)=(a_n+(b_n+c_n))=(a_n)+(b_n+c_n)=(a_n)+((b_n)+(c_n))$
- 1 works as expected $1(a_n)=(1a_n)=(a_n)$.
- And, it distributes $s((a_n)+(b_n))=(s(a_n+b_n))=(sa_n+sb_n)=(sa_n)+(sb_n)=s(a_n)+s(b_n)$.

Others would have more properties here, but for now we can be happy that Austin's list is shorter. Here's the other ones that we may have: $(s+t)\mathbf{v}=s\mathbf{v}+t\mathbf{v}$. and $s(t\mathbf{v})=(st)\mathbf{v}$.

I think the last part of the question is the most interesting. I hope you answered it. Hm, maybe it isn't the last question. But, is this a special case of functions from a nonempty set into $\mathbb{R}$? The answer is yes. Sequences can be thought of as functions from $\mathbb{N}$ to $\mathbb{R}$. We can consider $a:\mathbb{N}\to\mathbb{R}$ where $a(n)=a_n$, the nth term of the sequence. Of note, this is another infinite dimensional vector space. They sound fancy, but they are actually quite common in our experiences.

26. We want to know if A is a linear combination of the four given matrices, i.e. if

$$\begin{bmatrix}1&3\\0&0\end{bmatrix}=x\begin{bmatrix}1&0\\0&1\end{bmatrix}+y\begin{bmatrix}0&1\\1&0\end{bmatrix}+z\begin{bmatrix}1&0\\1&1\end{bmatrix}+w\begin{bmatrix}0&1\\1&1\end{bmatrix}$$

We can combine the right into one matrix to get $\begin{bmatrix}1&3\\0&0\end{bmatrix}=\begin{bmatrix}x+z&y+w\\y+z+w&x+z+w\end{bmatrix}$. Now, we know matrices are equal if their entries are equal so from here we have four equations:

$$1=x+z, 3=y+w, 0=y+z+w, 0=x+z+w$$

. Hey, it's a linear system, and this is the one thing we know best (or so I hope). So, what do we do, make a new matrix, 4×4, augmented: $\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$. I said I was done with sage, but I see no reason to do this by hand, this reduces to $\begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$. So the coefficients are $4, 4, -3, -1$, respectively. You may have seen this but doing this work makes the next problem much shorter than guessing this answer.

27. Now we want to show that these matrices span all of M^2, so instead of the matrix A above, we use a general $G = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. This changes our 4×4 augmented matrix to $\begin{bmatrix} 1 & 0 & 1 & 0 & a \\ 0 & 1 & 0 & 1 & b \\ 0 & 1 & 1 & 1 & c \\ 1 & 0 & 1 & 1 & d \end{bmatrix}$. The matrix row reduces to the identity as above. You don't need to find the coefficients to see that it will always be possible to do so. They will be some combination of a, b, c, d. Doing 26 well made 27 easier. And, maybe you wouldn't have thought to do this without seeing 26 first.

44. Starting question: are these four "vectors" independent? Is there a linear combination of them that equals zero? Well, let's try: $a(x^3 - x^2 + x + 1) + b(3x^3 + 2x + 1) + c(x^3 + 2x^2 - 1) + d(4x^3 + x^2 + 2x + 1) = 0$. Distribute and combine like terms to get $(a+3b+c+4d)x^3 + (-a+2c+d)x^2 + (a+2b+2d)x + (a+b-c+d) = 0$. And, here we go again. This gives us a linear system because two polynomials are equal when their coefficients are equal, so we have the linear system:

$$a + 3b + c + 4d = 0, -a + 2c + d = 0, a + 2b + 2d = 0, a + b - c + d = 0$$

And we're back to a 4×4 augmented matrix $\begin{bmatrix} 1 & 3 & 1 & 4 & 0 \\ -1 & 0 & 2 & 1 & 0 \\ 1 & 2 & 0 & 2 & 0 \\ 1 & 1 & -1 & 1 & 0 \end{bmatrix}$. Notice something that is convenient and valuable, the columns are the coefficients of the original polynomials. In fact, they *were* vectors all along! They are vectors in terms of the standard basis $\{x^3, x^2, x, 1\}$. And now I'm back to sage to row reduce for me: $\begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. So, there *is* a free variable, and the polynomials are not independent. We have the equations $a - 2c = 0$, $b + c = 0$, and $d = 0$. c is free. Letting $c = 1$ gives $a = 2$, $b = -1$ and so: $2(x^3 - x^2 + x + 1) + (-1)(3x^3 + 2x + 1) + 1(x^3 + 2x^2 - 1) = 0$, from this we have some choices, but what seems the most natural to me is that $(x^3 + 2x^2 - 1) = -2(x^3 - x^2 + x + 1) + 1(3x^3 + 2x + 1)$. So, that's our linear combination. (I presume you check something like that.). We also see that we have pivots in the first, second, and fourth columns, so those polynomials, i.e. $\{x^3 - x^2 + x + 1, 3x^3 + 2x + 1, 4x^3 + x^2 + 2x + 1\}$ are a basis for the subspace of $\mathbb{P}_3$.

62. It's too late for me to change, but I wish I hadn't included 62 and 63, as they would have been fine for our old vector spaces with vectors. These two questions are ... not good for us, let's say. Do please look at 64-67. $M = \left\{ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 \mid |u_1| \leq 4 \right\}$. Notice $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in M$, but it's not closed under scalar multiplication because $\begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix} \notin M$. Disappointing.

63. This one is not much better, but it is more annoying, but only because of pointless trigonometry. $M = \left\{ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 \mid \sin(u_1) = 1 \right\}$. Notice $\begin{bmatrix} \pi/2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in M$ because $\sin(\pi/2) = 1$, but it's not closed under scalar multiplication because $\begin{bmatrix} \pi \\ 0 \\ 0 \\ 0 \end{bmatrix} \notin M$ because $\sin \pi \neq 1$. Disappointing. Even more annoyingly useless than the last one. I am glad that no one is doing both. I'm pretty sure no one will pick this question. Ugh. Here's a much more solution to this one. The zero vector is not in this subset, so it cannot be a subspace since it isn't closed under multiplication by the real number zero.

Here's a bit more about why I don't like these questions. First of all, most simply, they do not generalise vector spaces, they use only boring $\mathbb{R}^4$. More than that, they only really use the first coordinate, since the condition only involves that coordinate. So, we're really looking at subspaces of $\mathbb{R}$ which is quite uninteresting. There are two possible subspaces for $\mathbb{R}$, zero and the entire line. These are neither of them. Please do remember that subspaces need to be geometrically flat. Subspaces of $\mathbb{R}^4$ can be: the point at zero, lines thru the origin, planes thru the origin, three-dimensional flat spaces thru the origin, and all of the space. Nothing else. Ok, I need to get going on the other questions.

64. Why do we call symmetric matrices W? I'm calling it S. $S \subset M^2 = \{A \mid A \in M^2, A^T = A\}$. This question is better. Is it closed under our operations? Given $A^T = A$, and $B^T = B$ is this also true for $A + rB$? Let's try. $(rA + B)^T = rA^T + B^T$ from properties of transpose, and from our givens this equals $A + rB$, so this is a good closed subspace. A simple question, and uses our ideas. How comforting.

65. This one takes a little more work, but still uses our ideas. We need to give many names to items this time. Here we go. Suppose $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $a + b = c + d$. That's one of them. We need another, let's use $K = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ and $x + y = z + w$. We also need a scalar, I'm sticking with r. The question then becomes: is $rM + K \in W$? To assess this we compute $rM + K = \begin{bmatrix} ra + x & rb + y \\ rc + z & rd + w \end{bmatrix}$. And now we check if the sum of the top row is equal to the sum of the bottom? The sum of the top is $(ra + x) + (rb + y) = r(a + b) + (x + y) = r(c + d) + (z + w) = (rc + z) + (rd + w)$, and that's what we want. Not so bad, but also on target. This is a subspace.

66. Now back to polynomials. suppose we have two polynomials in W, say one is $p = ax^3 + bx^2 + cx$, and another is $q = jx^3 + kx^2 + nx$. Notice they both have zero constant terms. Does that remain true if we compute $rp + q$ (where r remains a real number)? Let's check. $r(ax^3 + bx^2 + cx) + (jx^3 + kx^2 + nx) = (ra + j)x^3 + (rb + k)x^2 + (rc + n)x$. Hey, there's still not a constant term. So this. $rp + q \in W$. This is a subspace. This isn't difficult, but at least it's on topic.

67. We need one that doesn't work. Thankfully this is it. To show it *doesn't* work we need a counterexample. They are pretty easy to come by. If $p(2) = 1$ and $q(2) = 1$ then $(p + q)(2) = 1 + 1 = 2 \neq 1$. And if $p(2) = 1$, then $rp(2) = r \neq 1$ unless $r = 1$. Ok, that's great, now we just need a particular (notice the above is not *quite* enough, because there may be no such thing as what I've said above), thankfully *any* particular works. This is very much not a subspace. The easiest example I can think of is $p(x) = 1$, the constant polynomial. $2p(x) = 2$, so it is not closed under scalar multiplication. There are lots and lots of examples here. Once we got past the first two, the last questions here are reasonable.

Now for my two questions:

As a setup - from when you were in HS, what were linear polynomials? I'm guessing the answer here is $y = mx + b$ or $y = ax + b$.

Now, the real topic: consider some functions $T : \mathbb{P} \to \mathbb{P}$ (these are functions that take a polynomial to another polynomial): 1. take polynomial p and send it to $p(a)$ for a real number a. 2. take p and multiply by a polynomial q. 3. take p and add 1. Show all work to explain which ones are linear transformations from $\mathbb{P}$ to $\mathbb{P}$ and which are not. Are there any surprises compared to your original answer from HS?

For each we check if $T(rp+q) = rT(p) + T(q)$, where r is still a real number.

1. Here $T(p) = p(a)$. Let's see. $T(rp+q) = (rp+q)(a) = (rp)(a) + q(a) = r(p(a) + q(a) = rT(p) + T(q)$, so this *is* a linear transformation.

2. Here $T(p) = qp$ (polynomial is multiplication is commutative, it isn't wrong for it to be on the other side, but it feels more like T this way for me. The only tricky part is I need a new letter for a poly*n*omial. I'm picking $n(x)$. Now we want to know about $T(rp+n)$, so away we go: $T(rp+n) = q(rp+n) = qrp+qn = rqp+qn = rT(p)+T(n)$ which is what we want for a linear transformation, so this *is* a linear transformation.

3. This time $T(p) = p+1$. If we try to prove as above, it doesn't work, see: $T(rp+q) = rp+q+1$, but $rT(p)+T(q) = r(p+1+(q+1) = rp+q+(r+1)$. These are only equal if $r = 0$, where this is all just a silly way of writing $T(q) = T(q)$. Again, we need a counterexample to show this is real. Again they are extremely easy to find. As long as nothing is $r \neq 0$ we're ok. Let $p = 2$, $q = 3$ and $r = 4$ (these are all constant polynomials), then $rp+q = 11$. $T(rp+q) = 12$. On the other side, $rT(p)+T(q) = 4(3)+4 = 16 \neq 12$. Fine. So, unlike the HS interpretation, adding is *not* linear, but multiplying by *anything* is, as is substituting. Be careful about how you use language. Linear *Transformation* is special.

Ok, one more from me, and we'll be done done done!

Here is a transformation from $M^3 \to M^3$ (three by three matrices). Take matrix A and send it to $A^T + A$. Show that this is a linear transformation.

Let's call it T, so $T(A) = A^T + A$. So, now we check about $T(rA+B) = (rA+B)^T + (rA+B) = rA^T + B^T + rA + B = r(A^T + A) + (B^T + B) = rT(A) + T(B)$ and we're all set.

What is a basis for the kernel (same as nullspace) of this transformation? We want matrices such that $T(A) = \mathbf{0}$ i.e. where $A^T + A = \mathbf{0}$, or where $A^T = -A$. Ok, now - think. Think about taking a transpose. The diagonal doesn't move. Because of this in order for the diagonal elements, say $d = -d$ we need $d = 0$. So, the diagonal must be zero. The off-diagonals need to be negatives. So, one by one we make that. Here is a basis for the kernel: $\left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}$. There are other answers. This is the simplest one.

What is a basis for the image (= "range" or "column space")? This is the possible answers for $A^T + A$. Remember from an earlier problem set, $(A^T + A)^T = A^T + A$, these matrices are symmetric. And again we think. Think about what makes a symmetric matrix. First all diagonal matrices, and second those where off diagonals are in the same spaces. Following the above pattern we get matrices like those, but three more for the diagonals:

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}$$

What are the dimensions of each? Three and six respectively, count the basis. Notice that $3 + 6 = 9 = \dim(M^3)$.

With respect to Erin's basis (the standard basis) for M^3, what is the matrix of associated to this this transformation with this basis for both domain and target (="codomain"). As a reminder, here is the standard basis for M^3:

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

Our work is tedious now, but not difficult. We compute $T(B_i)$ for each $B_i \in$ basis. If you number them in order, I hope you can see the following $T(B_1) = 2B_1, T(B_2) = B_2 + B_4, T(B_3) = B_3 + B_7, T(B_4) = B_2 + B_4, T(B_5) = 2B_5, T(B_6) = B_6 + B_8, T(B_7) = B_3 + B_7, T(B_8) = B_6 + B_8, T(B_9) = 2B_9$. And each of these are used to form *column* vectors in our matrix for T: $\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$. It is a pretty matrix, I hope you agree. It is also symmetric, so the fact that the above work does give column vectors and not row vectors gives you a bit of grace here. Do be careful to recall that this technique of creating matrices in terms of a basis does produce *column* vectors in general. If you didn't like the "think" approach to finding bases above for subspaces, you could use our standard techniques on this 9×9 matrix to find the bases for subspaces. You would get the same answers. If someone *really* wants, I'll give $+2$ on this PS for anyone who brings this work with the 9×9 matrix to find the column and nullspaces in *on paper* to the final exam.