XM1 Soltuions

Here are some secrets that are deep in my philosophy of mathematics: mathematics without meaning is meaningless (hint: so is everything else). The whole point has *always* been about understanding. That has been true since you were small children, and it never stops being true. You have worked with others who lost track of that. I won't, and I'm glad to have David's book to support us in this quest. "Just tell me what to do and I will do it" isn't mathematics, that's what Sage is for. The same thing goes for "I know how to do it, I just don't know what it means." I very much recognise that this could be a drastically different way of thinking, but it is crucial, and I am not compromising on it.

Here's another truth: if you understand the concepts and language we use, the exam was straightforward. Here's the good news - there is not much *to* understand. But, doing it is essential. When you don't understand something we do in class - ask me, immediately. It is why we work in class where we are all together. The other good news here is that catching up (which is essential) is not difficult at this point. Soon it will be. Now, this weekend, before chapter 3 gets more serious, catch up. Make sure you understand the meanings.

Ok, that's preamble, let's get started. Please pull out the exam.

1. The key point here is to remember what pivots are. Pivots in a matrix are where there are leading ones *after* reducing the matrix to row echelon form. Look back at our first activity 1.4.1 on the topic. Once you understand that concept, this question should be quite straightforward. You want a matrix that will row reduce and have exactly two pivots. This means you want a matrix such that one off the three rows (because I said 3×3) is dependent on the others, and you want it so that the two other rows are independent. Beyond that, be as simple as possible, because you want to make the work easy for you. Yes, there's no reason for that, you could make all the numbers have fifteen digits, but what does that help? $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ has two dependent rows, obviously the first and the third and the second will reduce easily. Remember the question asks you to reduce, so ... make that as easy as you can for yourself. Think, think ahead. Open your eyes, don't squint. There are many many many right answers here, but making one that is easy for you is a valuable goal. This first example reduces to $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ in two easy steps. And then you see the two leading ones as desired.

Here's another similar example but that demonstrates a bit of the options, really all options are like one or the other, at least in their reduced form: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ which reduces to $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. All correct answers reduce to one of these forms.

2. This is the shortest question. The two bits of required information are what linear combination means and properties of matrix transformations. In order to write $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$ as a

linear combination of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$, we want a and b so that $\begin{bmatrix}4\\6\end{bmatrix} = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix}$. This really needs to *not* be something you "solve for", but something you look at and see. To get 4 in the first component we need $a = 4$ and to get 6 in the second we need $b = 6$ so $\begin{bmatrix}4\\6\end{bmatrix} = 4\begin{bmatrix}1\\0\end{bmatrix} + 6\begin{bmatrix}0\\1\end{bmatrix}$. Ideally you wrote far less than I have and just wrote the last equation, because it's simple. That's the first half of this very dull question. The second half uses the two properties: $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$ and $T(c\mathbf{v}) = cT(\mathbf{v})$. So, now we use the above linear combination (oh, by the way, there is only one possible answer for that, because the two simple vectors are independent [oh, understanding that matters, when I say things, understanding them matters]) and start with $T\left(\begin{bmatrix}4\\6\end{bmatrix}\right)$ and end with something that contains $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$, that's what "in terms of" means. That is also important. So, $T\left(\begin{bmatrix}4\\6\end{bmatrix}\right) = T\left(4\begin{bmatrix}1\\0\end{bmatrix} + 6\begin{bmatrix}0\\1\end{bmatrix}\right)$. We now use the first rule to separate the sum: $= T\left(4\begin{bmatrix}1\\0\end{bmatrix}\right) + T\left(6\begin{bmatrix}0\\1\end{bmatrix}\right)$ and next we use the second rule to move the constants out of T to get $= 4T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + 6T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$, and that's it. Look back at this question and see how simple it is *if* you understand our language. This is the power of understanding. In other matters, because of this very simple reason, this is why we focused so much on those two special vectors which we called $\mathbf{e}_1$ and $\mathbf{e}_2$ because knowing what the transformation does on them, we can find what it does to any vector using this simple technique because there was nothing special about 4 and 6.

3. Ok, here's something that you need to know - matrix transformations take vectors to vectors. We worked with them consistently in §2.5 an §2.6. You needed to be familiar and fluent with them. Matrix transformations have associated matrices, that's why we call them matrix transformations. $T(\mathbf{x}) = A\mathbf{x}$. For T to be "from $\mathbb{R}^2$ to $\mathbb{R}^3$" it takes two-dimensional vectors as inputs and produces three-dimensional vectors. For a matrix to be able to multiply by two-dimensional vectors, it needs to have two *columns*. For it to have three-dimensional outputs, it needs to have three *rows*. And all those thoughts need to be natural, because you are fluent in this language. This is not computation, this is meaning. So, out of that we need a 3×2 matrix *for* the transformation T. I said "in which no column is a multiple of another", but there are only two columns, so don't make them multiples of each other. There are no rules for zeroes this time. All you need are two columns with three entries each that are not multiples of each other. The simplest possible example is $\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}$. But there's no need to make it simple. Because the two vectors are not multiples, the span is a plane in $\mathbb{R}^3$. To be clear, it is *not* $\mathbb{R}^2$, which is two-dimensional vectors. That subtlety at the end here is the difference between stumbling over our language and speaking fluently. There are *lots and lots* of different possible examples here. The only thing needed is that the two columns are not multiples. All of them span planes on $\mathbb{R}^3$.

Short digression on how to give counterexamples. Here's the simplest example I know of to demonstrate this point. Suppose your professor makes you a promise "If you write an extra paper then you will get extra credit." What would make you angry? If you write the paper and *don't* get extra credit. You won't be angry if you don't write the paper but you get extra credit anyway, and you won't be (much) angry if you don't write the paper and don't get extra credit. The only way you are angry is if the first is true, but the second isn't. That's what we need for our counterexamples in question 4. It applies to both of them. Oh, and like I said on the exam O is a matrix with all zeroes.

4a. The statement is "if $A^2 = O$, then $A = O$", so we need an example where the first is true: $A^2 = O$, and the second is not, so $A \neq O$. A nonzero matrix that squares to a zero matrix. If you want to do this with 2×2, then there are really only two kinds of options one is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. The 1 can be any nonzero number and it could also be in the lower left. The second option is: $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. Any nonzero multiple of this works, as does rotating it. Interestingly those are the only options with 2×2. To be able to multiply by themselves the matrices must be square, but they could be bigger.

4b. The statement is "if $AC = AB$ and $A \neq O$, then $C = B$" So we want the first true, $AC = AB$ and $A \neq O$ and the second false, i.e. $C \neq B$. This one actually is more general than the first. If you let $A = C \neq O = B$, then you get the above solutions from 4a. Again make sure that you understand this. Here's a different kind of example, if you let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ then any two matrices with the same column sums will give the same result. I picked on $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & 3 \\ 6 & 3 \end{bmatrix}$. Both give $AB = AC = \begin{bmatrix} 4 & 6 \\ 4 & 6 \end{bmatrix}$. There are more options here on this one, but that gives you an idea. Perhaps most interesting are non-square options. That's fun, let me think about it a bit. If A is 2×3 then B and C need to be the same size as each other (but not as A) for $AC = AB$. Ok. Curious.

5a. I like this question a lot, mostly because the answers vary depending on your choice of a dependent set. If you have three dependent vectors and no two are multiples of each other, then removing any won't change the span. Here is the simplest example I can think of like that: $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. They are dependent as a sum, but because all have nonzero coefficients in that sum, we can solve for any one in terms of the others, so removing it won't change the span. This is *not* always true.

Here are two examples where it isn't true: $\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. The first two are multiples so interchangeable. Therefore, if we remove either of them we don't change anything, but removing the third will change the span into a line. One more example: $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. In this case, removing the zero vector doesn't change anything, but removing either of the others does.

5b. This one is different because it asks "can we always conclude?" To answer this we would need a reason why we can always conclude, or, we hope, the answer to be "no, because sometimes it happens differently." Not accidentally, the second is the case. I don't really know why the question is in $\mathbb{R}^3$, but it is. I would like to use $\mathbf{w} = \mathbf{v}$, but it says they must be different so I'll have $\mathbf{w} = 2\mathbf{v}$. So, here: $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$. They are all nonzero and different. $\mathbf{w}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$ precisely because $\mathbf{w} = 2\mathbf{v}$, but $\mathbf{u}$ is not in Span$\{\mathbf{v}, \mathbf{w}\}$ because Span$\{\mathbf{v}, \mathbf{w}\}$ is merely the y-axis and $\mathbf{u}$ is in the x-direction. There are fancier answers. Much like in 5a, the key here is that one can be a multiple of the others.

Epilogue: What do you take away from exam one? You need to understand our material, thoughtfully, to use our language comfortably. That requires thought, not rote practice. Talking intentionally helps - being precise in your language, and asking lots of questions helps. I promise I'm here to help you, but I can't do it if you don't talk to me. Reading is ok, talking is better. Here's a question – if you understand all that is here, is that enough? I happily believe the answer is *yes.**

*You *might* want to also be sure that you can write a parametric description of a solution to a system of equations. That skill will come in handy.