Mathmatics 239 solutions to Homework for Chapter 5

§24.4 Let A ={1, 2} and B = {3, 4}. Write down all the functions $f : A \to B$. Indicate which are two-to-two and which are onto B.

$f_1 = \{(1, 3), (2, 3)\}$ not two-to-two ($f(1) = f(2)$), not onto (there is no $f(a) = 4$).
$f_2 = \{(1, 3), (2, 4)\}$ two-to-two and onto.
$f_3 = \{(1, 4), (2, 3)\}$ two-to-two and onto.
$f_4 = \{(1, 4), (2, 4)\}$ not two-to-two ($f(1) = f(2)$), not onto (there is no $f(a) = 3$).

§24.8 Let $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7\}$. Let f be the relation

$$f = \{(1, 5), (2, 5), (3, 6), (?, ?)\}$$

Add an ordered pair so that

a) The relation f is not a function
$f = \{(1, 5), (2, 5), (3, 6), (3, 5)\}$
Not a function because $f(3) = 6$ and $f(3) = 5$.

b) The relation f is a function from A to B but is not onto B.
$f = \{(1, 5), (2, 5), (3, 6), (4, 5)\}$
Not onto because there is no $f(a) = 7$.

c) The relation f is a function from A to B and is onto B.
$f = \{(1, 5), (2, 5), (3, 6), (4, 7)\}$

EXTRA §24.11 Let a, b, and c be real numbers and consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax^2 + bx + c$. For which values of a, b, and c is f one-to-one? onto $\mathbb{R}$?

If $a \neq 0$, then the graph of $f(x)$ is a parabola. As a parabola $f(x)$ is not two-to-two because it is symmetric over its axis. Furthermore, as a parabola it has a vertex which is either the maximum (if $a < 0$) or minimum (if $a > 0$), and in either of those cases $f(x)$ is not onto because there will be no values greater than the maximum or less than the minimum.

So, we will assume $a = 0$. Once we make this assumption, if $b = 0$ then $f(x) = c$, a constant function, which is all-to-one and therefore extremely not onto.

Thus $f(x)$ is two-to-two and onto precisely when $a = 0$ and $b \neq 0$. There are no conditions on c. At this point the argument is now identical from class. If $f(x) = bx+c$ with $b \neq 0$, then we notice that $x = \frac{t-c}{b}$ goes to t for any $t \in \mathbb{R}$ so the function is onto, and if $bx + c = by + c$ then $x = y$ so the function is two-to-two.

EXTRA §24.13 This is a popular issue in Calc 2 (and in high school trigonometry). Technically $\sin x$ can't have an inverse, but it is *very* useful to ask the question "what angle has this sine value?" So, we make it so that we can answer this question by changing *both* the domain and the target. $\sin : \mathbb{R} \to \mathbb{R}$ does not have an inverse, but $\sin : [-\pi/2, \pi/2] \to [-1, 1]$ is a bijection, and this is the function that is inverted to produce $\sin^{-1}$.

§24.14 For each of the following, determine if the function is two-to-two, onto, neither or both. Prove your assertions.

a) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 2x$

This function is not onto; there is no integer x such that $2x = 3$.

This function is two-to-two. Proof:

Suppose that $f(x) = f(y)$. Then $2x = 2y$. Divide both sides by 2 to get $x = y$.

b) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 10 + x$.

This function is onto. Proof:

Suppose $z \in \mathbb{Z}. x = z - 10 \in \mathbb{Z}$ and $f(x) = 10 + (z - 10) = z$.

This function is two-to-two. Proof:

Suppose that $f(x) = f(y)$. Then $10 + x = 10 + y$. Subtract 10 from both sides to get $x = y$.

c) $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = 10 + x$.

This function is not onto; there is no natural number x such that $10 + x = 1$.

This function is two-to-two. Proof:

Suppose that $f(x) = f(y)$. Then $10 + x = 10 + y$. Subtract 10 from both sides to get $x = y$.

d) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{x-1}{2} & \text{if } x \text{ is odd} \end{cases}$

This function is onto. Proof:

Suppose $z \in \mathbb{Z}. x = 2z$ and $f(2z) = \frac{2z}{2} = z$ (note the first definition of f is used, as $2z$ is even, by definition)

This function is not two-to-two. $f(0) = f(1) = 0$.

e) $f : \mathbb{Q} \to \mathbb{Q}$ defined by $f(x) = x^2$.

This function is not onto; there is no rational number x such that $x^2 = 3$.

This function is not two-to-two, $f(1) = f(-1) = 1$.

§24.16 Let A and B be finite sets and let $f : A \to B$. Prove that any two of the following statements implies the third.

a) f is two-to-two

b) f is onto

c) $|A| = |B|$

a) $\wedge$ b) $\to$ c)

Prove: If f is two-to-two and f is onto, then $|A| = |B|$

If f is two-to-two and onto, then f is injective and surjective so f is bijective. Proposition 20.25 states that if f is a bijection on finite sets the sets have equal cardinality. Therefore $|A| = |B|$

a) $\wedge$ c) $\to$ b)

Prove: If f is two-to-two and $|A| = |B|$, then f is onto.

We will prove this by contradiction. Suppose that f is two-to-two and $|A| = |B|$. For the sake of contradiction assume that f is not onto. If f is not onto, then there is some element in B that is not in $\text{Im}(f)$ [recall $\text{Im}(f)$ is the set of actual values that f takes. It is a subset of the target, here B]. Therefore $|\text{Im}(f)| < |B|$. But $|A| = |B|$. Therefore $|\text{Im}(f)| < |A|$. Therefore the number of values that f takes is less than the number of domain elements. Therefore by the pigeonhole principle, f cannot be an injection. But we assumed that f is two-to-two. This is a contradiction. Therefore we must deny our last assumption, namely that f is not onto. Therefore we have shown that f must be onto, as desired.

b) $\wedge$ c) $\rightarrow$ a)

Prove: If f is onto and $|A| = |B|$ then f is two-to-two. We will prove this by contradiction. Suppose that f is onto and $|A| = |B|$. For the sake of contradiction assume that f is not two-to-two. If f is not two-to-two, then there is some pair of elements in A that go to the same element in B. Therefore $|\mathrm{Im}(f)| < |A|$ (because of the overlapping). But $|A| = |B|$. Therefore $|\mathrm{Im}(f)| < |B|$. Therefore the number of values that f takes is less than the number of target elements. Therefore, f must miss some target elements, so f cannot be a surjection. But we assumed that f is onto. This is a contradiction. Therefore we must deny our last assumption, namely that f is not two-to-two. Therefore we have shown that f must be two-to-two, as desired.

§24.17 Give an example of a set A and a function $f : A \rightarrow A$ where f is onto but not two-to-two. - See §24.14 d)

Give an example where f is one-to-one and not onto. - See §24.14 a)

Are your examples contradictions to the previous exercise? No, both have $A = \mathbb{Z}$ which is an infinite set. §24.16 proved a result that holds only for finite sets.

§24.20 Let A be an n-element set and let $k \in \mathbb{N}$. How many functions $f : A \rightarrow \{0, 1\}$ are there for which there are exactly k elements a in A with $f(a) = 1$?

Creating such a function corresponds to choosing the subset of A that has $f(a) = 1$. So, it suffices to count the possible subsets of A with k elements. By definition there are $\binom{n}{r}$ of these.

§24.22 Let A be an n-element set and let $i, j, k \in \mathbb{N}$ with $i + j + k = n$. How many functions $f : A \rightarrow \{0, 1, 2\}$ are there for which there are exactly i elements $a \in A$ with $f(a) = 0$, exactly j elements $a \in A$ with $f(a) = 1$ and exactly k elements $a \in A$ with $f(a) = 2$. This is an extension of the previous problem. First choose the elements to go to 0 in $\binom{n}{i}$ ways, then choose the elements to go to 1 in $\binom{n-i}{j}$ ways. The rest go to 2, or alternatively we have the rest in $\binom{n-i-j}{k} = \binom{k}{k} = 1$ way. Altogether we have $\binom{n}{i}\binom{n-i}{j}$ such functions. Computing this out yields $\frac{n!}{i!j!k!} = \binom{n}{i\ \ j\ \ k}$. See exercises §17.29-31 for more on this topic.

EXTRA §24.23 $f(X) = \{f(x) \mid x \in X\}$

a. $f(X) = \{f(-1), f(0), f(1), f(2)\} = \{0, 1, 2\}$ (1 need not be listed twice.)

b. Considering a graph of $\sin(x)$, notice that it starts at 0, goes up to 1 and returns to 0 at π. So $f([0, \pi]) = [0, 1]$.

c. Since $f(x)$ is monotonic and $f(-1) = \frac{1}{2}$ and $f(1) = 2$, $f([-1, 1]) = [\frac{1}{2}, 2]$.

d. $f(\{1\}) = \{f(1)\} = \{2\}$, which is not equal to $f(1) = 2$, since one is a set and one is a number.

e. $f(A) = \text{im}(f)$.

EXTRA §24.24 a. $f : \mathbb{Z} \to \mathbb{Z}$ where $x \mapsto |x|$. $f^{-1}(\{1,2,3\}) = \{-3,-2,-1,1,2,3\}$.
b. $f : \mathbb{R} \to \mathbb{R}$ where $x \mapsto x^2$. $f^{-1}([1,2]) = [-\sqrt{2},-1] \cup [1,\sqrt{2}]$.
c. $f : \mathbb{R} \to \mathbb{R}$ where $x \mapsto \frac{1}{1+x^2}$ $f^{-1}(\{\frac{1}{2}\}) = \{-1,1\}$
d. $f : \mathbb{R} \to \mathbb{R}$ where $x \mapsto \frac{1}{1+x^2}$ $f^{-1}(\{-\frac{1}{2}\}) = \emptyset$ because this function is always positive, so never negative.

§25.9 Consider a square whose side length has length one. Suppose we select five points from this square. Prove there are two points whose distance is at most $\frac{\sqrt{2}}{2}$.

First divide the square into four smaller squares. If we distribute five points into these four squares by the pigeonhole principle there will be one small square with at least two points. The maximal distance between two points in a small square is $\frac{\sqrt{2}}{2}$. Therefore there will be two points with distance at most $\frac{\sqrt{2}}{2}$.

EXTRA §25.10 The proof of 25.2 tells us exactly how to do this. We want to select one each from the "types" given in the proof. So, here we go: (even, even) = (0,0), (even,odd) = (0,1), (odd, even) = (1,0), and (odd, odd) = (1,1). These are vertices of a square. Notice in particular that there are no lattice points contained in the square except for these corners. The midpoints are not at the corners, so they are no lattice points. We could compute all six of them, but this justification shows that certainly none of them will be lattice points.

EXTRA §25.15 This uses the increasing / decreasing sequence work we discussed. Consider the dots from left to right. Their heights could be measured, and therefore could form a sequence of $10 = 3^2 + 1$ numbers. In a sequence of $3^2 + 1$ numbers there is a monotonic subsequence of length $3 + 1 = 4$. So, there are four points that are either increasing or decreasing, i.e. that make either an upward or downward path.

§25.16 Let $f : N \to Z$ by $f(n) = \begin{cases} -\frac{n}{2} & \text{if } n \text{ is even and} \\ \frac{(n+1)}{2} & \text{if } n \text{ is odd.} \end{cases}$ Prove that f is a bijection.

First: f is surjective. Please notice, cases must be based on the *output* not on the input.

Suppose there is a $z \in \mathbb{Z}$. Find n by this plan $n = \begin{cases} -2z & \text{if } z \leq 0 \\ 2z-1 & \text{if } z > 0 \end{cases}$. Consider $f(n)$. When $n = -2z$, n is even so $f(n) = -\frac{(-2z)}{2} = z$. When $n = 2z-1$, n is odd so $f(n) = \frac{(2z-1)+1}{2} = z$.

Next: f is injective.

Suppose $f(n) = f(m)$. Before proceeding, note that if n is a natural number, $\frac{-n}{2}$ is always nonpositive and $\frac{n+1}{2}$ is always positive. We will therefore break this into two non-overlapping cases again based on the *output*.

Case 1: Suppose $f(n) = f(m) \leq 0$. Then $-\frac{n}{2} = -\frac{m}{2}$. Multiply both sides by -2 to get $n = m$.

Case 2: Suppose $f(n) = f(m) > 0$. Then $\frac{n+1}{2} = \frac{m+1}{2}$. Multiply both sides by 2, then subtract 1 from both sides to get $n = m$. Therefore: if $f(n) = f(m)$ then $n = m$. Hence f is injective. As f is both injective and surjective it is bijective, as desired.

§26.4 What is the difference between the identity function defined on a set A and the is-equal-to relation defined on A?

Let us consider an example. Consider $A = \{x, y, z\}$. Then as ordered pairs, the is-equal to relation is "=" $= \{(x, x), (y, y), (z, z)\}$. The identity function is $\iota = \{(x, x), (y, y), (z, z)\}$. So as ordered pairs they are the same. However, we do not use them in the same way. We think of the relation as producing true or false statements about comparison and we think of the function as a rule for taking elements of A to other elements of A. These two objects are very similar in formality but practically are very different. By analogy this is akin to two people using the same tool for very different purposes. We use the same set for ι and $=$ but we use them for very different purposes.

§26.12 Suppose $f : A \to A$ and $g : A \to A$ are both bijections.

a) Prove or disprove: $g \circ f$ is a bijection from A to itself.

First: $g \circ f$ is surjective

Suppose $a \in A$. Because g is surjective there is a $b \in A$ such that $g(b) = a$. Because f is surjective there is a $c \in A$ such that $f(c) = b$. Consider $(g \circ f)(c) = g(f(c)) = g(b) = a$. Therefore there is a $c \in A$ such that $(g \circ f)(c) = a$. Hence $g \circ f$ is surjective.

Second: $g \circ f$ is injective

Suppose $(g \circ f)(x) = (g \circ f)(y)$. Then $g(f(x)) = g(f(y))$. Because g is injective, $f(x) = f(y)$. Because f is injective $x = y$. Therefore we have shown than if $(g \circ f)(x) = (g \circ f)(y)$ then $x = y$. Hence $g \circ f$ is injective.

As $g \circ f$ is both injective and surjective it is bijective, as desired.

b) Prove or disprove: $(g \circ f)^{-1} = g^{-1} \circ f^{-1}$.

This is false. Consider $A = \{1, 2, 3\}$.

Let $f = \{(1, 3), (2, 1), (3, 2)\}$ and $g = \{(1, 2), (2, 1), (3, 3)\}$

Then $f^{-1} = \{(1, 2), (2, 3), (3, 1)\}$ and $g^{-1} = \{(1, 2), (2, 1), (3, 3)\}$.

$g \circ f = \{(1, 3), (2, 2), (3, 1)\}$

$(g \circ f)^{-1} = \{(1, 3), (2, 2), (3, 1)\}$

$g^{-1} o f^{-1} = \{(1, 1), (2, 3), (3, 2)\}$

Therefore $(g \circ f)^{-1} \neq g^{-1} \circ f^{-1}$.

In cycle notation $f = (1, 2, 3)$ and $g = (1, 2)$.

$f^{-1} = (1, 3, 2)$, $g^{-1} = (1, 2)$

$g \circ f = (1, 3)$

$(g \circ f)^{-1} = (1, 3)$

$g^{-1} \circ f^{-1} = (2, 3)$

Therefore $(g \circ f)^{-1} \neq g^{-1} \circ f^{-1}$.

c) Prove or disprove: $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

First consider the domain of the two functions:

The domain of $(g \circ f)^{-1}$ is the target of $g \circ f$ which is the target of g which is A.

The domain of $f^{-1} \circ g^{-1}$ is the domain of g^{-1} which is the target of g which is A.

Therefore they have the same domain.

Let $a \in A$. Consider $(g \circ f)^{-1}(a)$ and $(f^{-1} \circ g^{-1})(a)$. Let us evaluate the second first. $(f^{-1} \circ g^{-1})(a) = f^{-1}(g^{-1}(a))$. Using the above notation we have $g(b) = a$, so $g^{-1}(a) = b$.

Therefore $(f^{-1} \circ g^{-1})(a) = f^{-1}(b)$. Again using the above notation we have $f(c) = b$, so $f^{-1}(b) = c$. giving $(f^{-1} \circ g^{-1})(a) = c$.

We know that $(g \circ f)(c) = g(f(c)) = g(b) = a$, so $(g \circ f)(c) = a$. Hence $(g \circ f)^{-1}(a) = c$. Therefore we have $(g \circ f)^{-1} = c = f^{-1} \circ g^{-1}$, as desired.

§27.2 Please express the following permutations in disjoint cycle form.

a) $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 1 & 3 & 5 \end{bmatrix} = (1,2,4)(3,6,5)$

b) $\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{bmatrix} = (1,2,3,4,5,6)$

c) $\pi \circ \pi = (1,2,3,4,5,6)(1,2,3,4,5,6) = (1,3,5)(2,4,6)$

d) $\pi^{-1} = (1,6,5,4,3,2)$

e) $\iota =$ either the empty cycle or $(1)(2)(3)(4)(5)$

f) $(1,2)(2,3)(3,4)(4,5)(5,1) = (1)(2,3,4,5) = (2,3,4,5)$

g) $\{1,2),(2,6),(3,5),(4,4),(5,3),(6,1)\} = (1,2,6)(3,5) = (1,2,6)(3,5)(4)$.

§27.13 Let $\pi = (1,2)(3,4,5,6,7)(8,9,10,11)(12) \in S_{12}$. Find the smallest positive integer k so that $\pi^{(k)} = \iota$. Generalize.

The cycle (1, 2) will be the identity with two applications of π, and with any multiple of two applications of π. Similarly the cycle (3, 4, 5, 6, 7) will be the identity with five applications of π, and with any multiple of five applications of π. The cycle (8, 9, 10, 11) will be the identity with four applications of π and any multiple of four, and the cycle (12) is the identity already. So, for the entire permutation to be the identity, we need it to be applied a number of times that is a multiple of 2, 4, 5 and 1. The least multiple of all of these numbers is 20. Therefore 20 is the least k such that $\pi^{(k)} = \iota$.

By the same reasoning, for any permutation written with disjoint cycles of lengths $n_1, n_2, \ldots, n_t$, the smallest integer k so that $\pi^{(k)} = \iota$ is the least common multiple of $n_1, n_2, \ldots, n_t$.

Extra problems

1. Prove Let A and B be two sets. If there exist one-to-one functions $f : A \to B$ and $g : B \to A$, then there exists a function $h : A \to B$ that is a bijection.

This is the Schroeder-Bernstein Theorem. It is not easy to prove - you can look for proofs on-line. Again, I'll be happy to discuss this with you.

2. Describe a bijection from the closed interval [0, 1] onto the half-closed interval [0, 1).

This is an application of the SB Theorem. It's easy to have injections both ways here - simply divide by two. Then apply the ideas in SB.

§27.21 Sam Loyd's Fifteen Puzzle is a 4×4 array of tiles numbered 1 to 15 with one empty space. You move the tiles about this board by sliding a number tile into the empty position. The initial configuration is shown first. The object of the puzzle is to use a sequence of legal moves to switch the positions of tiles 14 and 15 while returning all other tiles to their original positions. (In fact, when Loyd introduced the puzzle in the 1870s, there was a prize of \$1,000, offered for the first correct solution to the problem, has never been claimed, although there are thousands of persons who say they performed the required feat.)

The reason none claimed Loyd's prize (and presumeably the reason he offered it in the first place) is that the goal is impossible to attain. Consider the puzzle as representing permutations of the numbers 1 to 15 and the blank square. Clearly the desired goal represents a transposition of the original configuration (by switching one pair, (14, 15)). Therefore, it should be performed by an odd permutation.

On the other hand, the goal includes returning the blank square to its starting space. To move the move the blank square away and back requires an even number of transpositions (consider that the blank must move the same number of squares left as right and the same number of squares up as down). Therefore, the goal should be performed by an even permutation. The desired permutation would therefore be both even and odd, however we have proven that a permutation may not be both even and odd. Therefore, there is no permutation that performs the desired move.