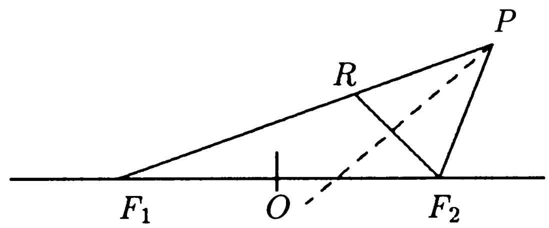

FIGURE 1.11. Exercise 11.

12. Using the results of Exercise 11, prove that the slope of RF_2 is

$$\frac{t}{s-c} = \frac{-b}{a}\cot\theta,$$

thus proving Lemma 1.6.

13. In Figure 1.10, extend the line through S, X, and V so that it intersects the line PK at W. Let $\theta = \angle RPS$ and $\phi = \angle F_1PK$. Show that $|SX|/|SW| = 1 - \tan\theta\tan\phi$. As S approaches P, ϕ stays fixed but θ approaches 0. Use this to conclude that $|SX|/|SV|$ approaches 1.

14. The quote from *Harmonice Mundi* is something of a cheat. It only refers to the third law, the first two having been published ten years earlier in *Astronomia Nova.* Kepler also had a fourth law governing the relative distances of the planets from the sun. It is now conveniently forgotten by most scientists since it was wrong. What was Kepler's fourth law? Hint: see Figure 1.12, taken from his book *Mysterium Cosmographicum*, published in 1596.

1.5 Reprise with Calculus

While I find Newton's proof of Theorem 1.1 very appealing in its simplicity and the clarity with which it illuminates the connection between radial force and equal areas, his proof of Theorem 1.3 is not as transparent as one would wish. There is much more to Newton's *Principia.* He goes on to derive Kepler's laws from the law of gravity and then to explore the consequences of his insights. The entirety of *Principia* consists of three volumes. But, at this point, I want to leave Newton and find a simpler language for explaining celestial mechanics.

The search for a better idiom in which to understand our mathematics is going to be a recurrent theme throughout this book. It is not always easy to make the transition; new concepts are often at a high level of

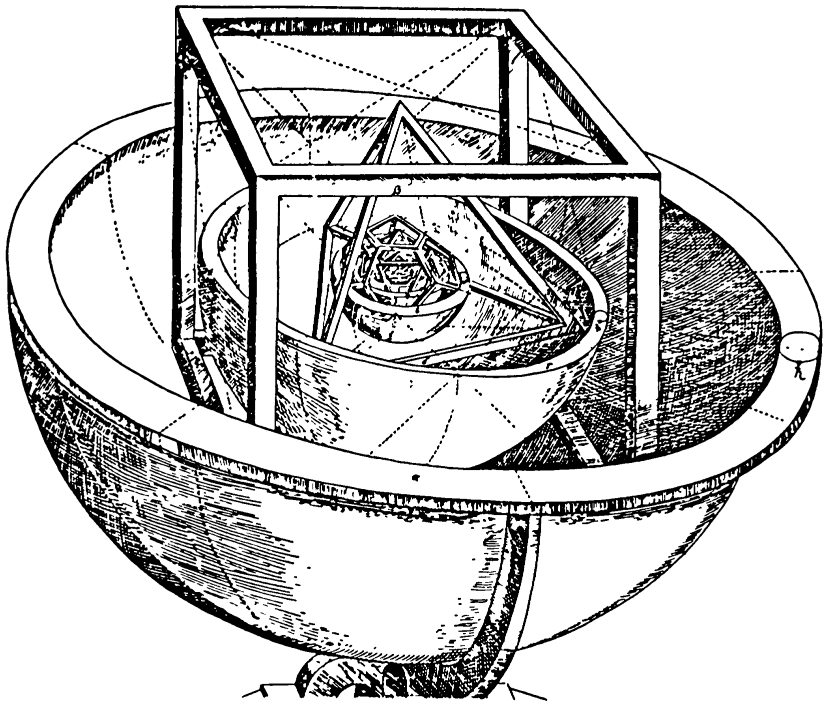

FIGURE 1.12. Kepler's fourth law?

abstraction and difficult to grasp. But, once you are comfortable with the new terminology, it can greatly clarify relationships and proofs. I feel that the effort expended is more than repaid in a better understanding of the material at hand and an enhanced ability to build on it. It is worth keeping in mind, however, that we always have choices and that future generations may look upon our expressions and proofs as unnecessarily convoluted.

Trajectories as Functions

In moving into the language of calculus, we first need to describe the trajectory of our moving particle as a function. We can think of its position $\vec{r}(t)$ as a function of time. For this chapter, we shall stay in the x, y plane. The x and y coordinates are each functions of time:

$$\vec{r}(t) = (x(t), y(t)).$$

For any specific value of t, we can think of $\vec{r}(t)$ as *either a point in the plane or as the vector from the origin to this point.* While initially somewhat confusing, it is very convenient to be able to move freely between these two interpretations.

We speak of $\vec{r}(t)$ as a function from one real variable, t, to two real

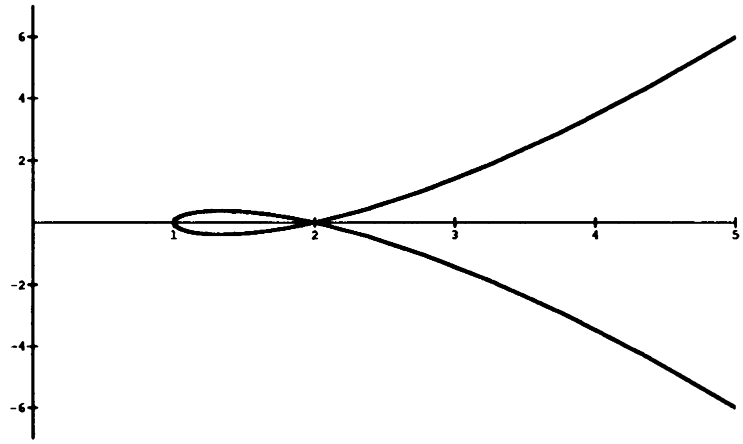

FIGURE 1.13. $\vec{r}(t) = (t^2 + 1, t^3 - t)$.

variables, x and y. This aspect of $\vec{r}$ is described in notational shorthand:

$$\vec{r} : \mathbf{R} \longrightarrow \mathbf{R}^2.$$

A function from one variable to one variable such as $f(t) = t^3 - t$ is called a *scalar function.* A function from one variable to more than one variable such as $\vec{r}(t)$ is called a *vector function.* We shall restrict our attention to functions for which the derivative is defined at all or almost all values of t.

A nice example is the path (Figure 1.13) for which the position at time t is given by

$$\begin{aligned} x(t) &= t^2 + 1, \\ y(t) &= t^3 - t. \end{aligned}$$

The velocity vector at time t, $\vec{v}(t)$, is the rate at which the position is changing:

$$\vec{v} = \frac{d\vec{r}}{dt},$$

and is uniquely determined by the rate at which the x coordinate of our particle is changing:

$$\frac{dx}{dt} = 2t$$

and the rate at which the y coordinate is changing

$$\frac{dy}{dt} = 3t^2 - 1,$$

so that we have for our example

$$\vec{v}(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (2t, 3t^2 - 1).$$

Here is where we can exploit the ambiguity between points and vectors. It is natural to think of the velocity as a vector. At time $t = 2$, our particle is at the point $(5, 6)$ moving with a velocity $(4, 11)$, that is, the velocity has the same magnitude and direction as the vector from $(0, 0)$ to $(4, 11)$. But, we can also view the velocity as a vector function describing its own path in the x, y plane and ask how the velocity is changing. The rate of change of the velocity is the acceleration $\vec{a}(t)$, which is the derivative of the velocity:

$$\vec{a} = \frac{d\vec{v}}{dt} = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}\right) = (2, 6t).$$

For the problem at hand, that of understanding celestial mechanics, it is easiest to use the polar coordinates $r(t)$, the distance to the origin at time t, and $\theta(t)$, the angle between $\vec{r}(t)$ and the positive x axis. We shall need to assume that we stay away from the origin so that $r(t)$ is never zero. Note that $r(t)$ is the magnitude of the vector $\vec{r}(t)$:

$$r(t) = |\vec{r}(t)|. \tag{1.6}$$

The relationships between rectangular and polar coordinates are given by

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{1.7}$$

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = y/x. \tag{1.8}$$

Local Coordinates

It is also convenient to compute in terms of *local coordinates* that change as our particle moves. In particular, we shall want to decompose the acceleration into a component parallel to the vector $\vec{r}(t)$ and a second component perpendicular to it. This can be done by defining a *unit vector* or vector of length 1 in the direction of $\vec{r}$ by

$$\vec{u}_r(t) = \frac{\vec{r}(t)}{r(t)} = \frac{(r(t)\cos\theta(t), r(t)\sin\theta(t))}{r(t)} = (\cos\theta(t), \sin\theta(t)) \tag{1.9}$$

and a perpendicular unit vector (Figure 1.14)

$$\vec{u}_\theta(t) = (-\sin\theta(t), \cos\theta(t)). \tag{1.10}$$

It is important to keep in mind that $\vec{u}_r$ and $\vec{u}_\theta$ are functions of t. In particular, they have derivatives that are related:

$$\frac{d\vec{u}_r}{dt} = \left(-\sin\theta\,\frac{d\theta}{dt}, \cos\theta\,\frac{d\theta}{dt}\right) = \frac{d\theta}{dt}\,\vec{u}_\theta, \tag{1.11}$$

$$\frac{d\vec{u}_\theta}{dt} = \left(-\cos\theta\,\frac{d\theta}{dt}, -\sin\theta\,\frac{d\theta}{dt}\right) = -\frac{d\theta}{dt}\,\vec{u}_r. \tag{1.12}$$

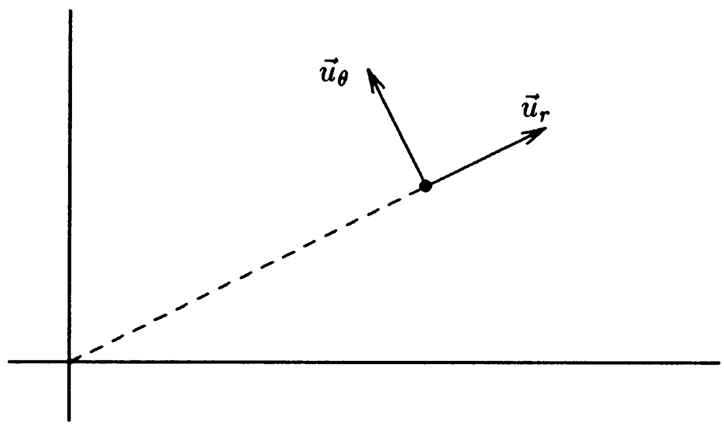

FIGURE 1.14. The local coordinates $\vec{u}_r$ and $\vec{u}_\theta$.

Since the product rule for derivatives works on each coordinate, it also holds for the product of a scalar function times a vector function:

$$\frac{d}{dt}(f(t)\vec{r}(t)) = \frac{df}{dt}\vec{r}(t) + f(t)\frac{d\vec{r}(t)}{dt}. \tag{1.13}$$

Combining these results with the fact that

$$\vec{r} = r\vec{u}_r, \tag{1.14}$$

we see that

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\,\vec{u}_r) = \frac{dr}{dt}\,\vec{u}_r + r\,\frac{d\vec{u}_r}{dt} = \frac{dr}{dt}\,\vec{u}_r + r\frac{d\theta}{dt}\,\vec{u}_\theta, \tag{1.15}$$

$$\begin{aligned}
\vec{a} &= \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{dr}{dt}\,\vec{u}_r\right) + \frac{d}{dt}\left(r\frac{d\theta}{dt}\,\vec{u}_\theta\right) \\
&= \frac{d^2r}{dt^2}\,\vec{u}_r + \frac{dr}{dt}\frac{d\theta}{dt}\,\vec{u}_\theta + \frac{dr}{dt}\frac{d\theta}{dt}\,\vec{u}_\theta + r\,\frac{d^2\theta}{dt^2}\,\vec{u}_\theta + r\,\frac{d\theta}{dt}\left(-\frac{d\theta}{dt}\,\vec{u}_r\right) \\
&= \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\vec{u}_r + \left(r\,\frac{d^2\theta}{dt^2} + 2\,\frac{dr}{dt}\frac{d\theta}{dt}\right)\vec{u}_\theta \\
&= \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\vec{u}_r + \frac{1}{r}\frac{d}{dt}\left(r^2\,\frac{d\theta}{dt}\right)\vec{u}_\theta.
\end{aligned} \tag{1.16}$$

Modern Proof of Kepler's First Law

Equation (1.16) tells us that acceleration is entirely radial, that is, parallel to $\vec{r}$, if and only if

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0,$$

which is equivalent to saying that $r(t)^2\,d\theta/dt$ is a constant independent of t. We have proven the following result.

Lemma 1.8 *If the position of a particle over time is described by the vector function $\vec{r}(t)$, then the acceleration is purely radial if and only if $r(t)^2\,d\theta/dt$ is a constant independent of t.*

Combining this with the next lemma gives us another proof of Theorems 1.1 and 1.2.

Lemma 1.9 *If the position of a particle over time is described by the vector function $\vec{r}(t)$, then the rate at which the radial vector sweeps out area is given by*

$$\frac{dA}{dt} = \frac{1}{2}r(t)^2\,\frac{d\theta}{dt}. \qquad (1.17)$$

Proof: Given a circle with center at the origin and radius r, the area of the sector swept out by the radius as it moves through an angle of $\Delta\theta$ is given by $(r^2/2)\Delta\theta$. It follows that if ΔA is the area swept out by the radial vector from time s to time t and if the distance from the origin stays constant during this time interval, then

$$\Delta A = \frac{r^2}{2}\,\Delta\theta,$$

where $\Delta\,\theta = \theta(t) - \theta(s)$.

If r does not stay constant, then we can find two points in the interval $[s, t]$, call them t_1 and t_2, where r takes on its minimum and maximum values, respectively, over this interval:

$$r(t_1) \le r \le r(t_2).$$

It follows that

$$\frac{r(t_1)^2}{2}\Delta\theta \le \Delta A \le \frac{r(t_2)^2}{2}\Delta\theta.$$

We now divide through by $\Delta t = t - s$:

$$\frac{r(t_1)^2}{2}\,\frac{\Delta\theta}{\Delta t} \le \frac{\Delta A}{\Delta t} \le \frac{r(t_2)^2}{2}\,\frac{\Delta\theta}{\Delta t}$$

and take the limit as s approaches t. This forces t_1 and t_2 to also approach t and yields

$$\frac{r(t)^2}{2}\frac{d\theta}{dt} \le \frac{dA}{dt} \le \frac{r(t)^2}{2}\frac{d\theta}{dt}.$$

Q.E.D.

Modern Proof of the Law of Gravity

We shall now use calculus to demonstrate that if the acceleration is purely radial and the path is an ellipse with one focus at the origin, then the acceleration is inversely proportional to the square of the distance from the origin.

Lemma 1.10 *The general equation of an ellipse with one focus at the origin and major axis on the x axis is given in polar coordinates by*

$$r(1 + \varepsilon \cos\theta) = c, \tag{1.18}$$

where ε and c are real constants, $|\varepsilon| < 1$, $c > 0$. The semimajor axis is $c/(1-\varepsilon^2)$ and the semiminor axis is $c/\sqrt{1-\varepsilon^2}$.

Proof: Equation (1.18) can be rewritten as

$$r = c - \varepsilon r \cos\theta.$$

We use Equations (1.7) and (1.8) to convert this to rectangular coordinates and then square each side to obtain

$$x^2 + y^2 = c^2 - 2c\varepsilon x + \varepsilon^2 x^2,$$

which can be rewritten as

$$\begin{aligned}(1-\varepsilon^2)\left(x^2 + \frac{2c\varepsilon x}{1-\varepsilon^2} + \frac{c^2\varepsilon^2}{(1-\varepsilon^2)^2}\right) + y^2 &= c^2 + \frac{c^2\varepsilon^2}{1-\varepsilon^2} \\ &= \frac{c^2}{1-\varepsilon^2}, \\ \frac{(x + c\varepsilon/(1-\varepsilon^2))^2}{a^2} + \frac{y^2}{b^2} &= 1, \qquad (1.19)\end{aligned}$$

where $a = c/(1-\varepsilon^2), b = c/\sqrt{1-\varepsilon^2}$.

The center of this ellipse is at $(-c\varepsilon/(1-\varepsilon^2), 0)$ which is $c|\varepsilon|/(1-\varepsilon^2)$ from the origin. This is precisely the distance of the focus from the center of the ellipse:

$$\sqrt{a^2 - b^2} = \sqrt{\frac{c^2}{(1-\varepsilon^2)^2} - \frac{c^2}{1-\varepsilon^2}} = \sqrt{\frac{c^2\varepsilon^2}{(1-\varepsilon^2)^2}} = \frac{c|\varepsilon|}{1-\varepsilon^2}.$$

Q.E.D.

Theorem 1.4 *If a particle moves so that its acceleration is always radial and if the particle follows the curve given by Equation (1.18), then the acceleration is*

$$\vec{a}(t) = \frac{-k^2}{cr^2}\vec{u}_r, \tag{1.20}$$

where $k = 2\,(dA/dt)$ is a constant.

Proof: The fact that dA/dt is constant follows from Theorem 1.2. From Lemma 1.9 we know that

$$r^2\frac{d\theta}{dt} = k. \tag{1.21}$$

If we solve for $d\theta/dt$:

$$\frac{d\theta}{dt} = kr^{-2}, \tag{1.22}$$

then we can substitute into our expression for the acceleration [Equation (1.16)]:

$$\vec{a} = \left(\frac{d^2r}{dt^2} - \frac{k^2}{r^3}\right)\vec{u}_r. \tag{1.23}$$

We solve Equation (1.18) for r and then differentiate with respect to t:

$$r = \frac{c}{1 + \varepsilon\cos\theta},$$

$$\frac{dr}{dt} = \frac{-c\varepsilon(-\sin\theta)}{(1+\varepsilon\cos\theta)^2}\frac{d\theta}{dt} = \frac{c\varepsilon\sin\theta}{c^2r^{-2}}\frac{k}{r^2},$$

where the last equality uses Equations (1.18) and (1.22). Simplifying this expression, we obtain

$$\frac{dr}{dt} = \frac{k\varepsilon}{c}\sin\theta.$$

We differentiate a second time and again use Equations (1.18) [in the form $\cos\theta = (c-r)/r\varepsilon$] and (1.22) to simplify:

$$\frac{d^2r}{dt^2} = \frac{k\varepsilon}{c}\cos\theta\frac{d\theta}{dt} = \frac{k\varepsilon}{c}\frac{(c-r)}{r\varepsilon}\frac{k}{r^2} = \frac{k^2}{r^3} - \frac{k^2}{cr^2},$$

and therefore

$$\vec{a} = \left(\frac{d^2r}{dt^2} - \frac{k^2}{r^3}\right)\vec{u}_r = -\frac{k^2}{cr^2}\vec{u}_r. \tag{1.24}$$

Q.E.D.

1.6 Exercises

1. For each of the following paths given by $\vec{r}(t)$, find the position of the particle at the specified times and sketch the path.

   (a) $\vec{r}(t) = (3\cos t, \sin t), t = 0, \pi/3, 3\pi/4, 3\pi/2.$
   (b) $\vec{r}(t) = (\sin t, t^2 - 1), t = -\pi/2, 0, \pi/2, \pi.$
   (c) $\vec{r}(t) = (\sqrt{t}, \sqrt{t}/(t+1)), t = 1/4, 1, 2, 4.$
   (d) $\vec{r}(t) = (t - 2\sin t, 1 - 2\cos t), t = -\pi/3, 0, \pi, 2\pi.$

2. For each of the vector functions in Exercise 1, find the velocity $\vec{v}$ at the indicated times.

3. For each of the vector functions in Exercise 1, find the acceleration $\vec{a}$ at the indicated times.

4. Using the relationships of Equation (1.7), prove that

$$\frac{d\theta}{dt} = \frac{x\,(dy/dt) - y\,(dx/dt)}{x^2 + y^2}. \tag{1.25}$$

   It follows that acceleration is radial if and only if $x\,(dy/dt)$ and $y\,(dx/dt)$ differ by a constant.

   In Exercises 5–10, let $\vec{r}(t) = (t^2 - t, t\sqrt{2t - t^2}), 0 \le t \le 2$.

5. Sketch the curve described by $\vec{r}(t)$.

6. Find $r(t)$.

7. Find $\vec{u}_r$ and $\vec{u}_\theta$ as functions of t.

8. Find dr/dt and $d\theta/dt$. (Hint: use Exercise 4.)

9. Express the velocity in terms of the local coordinates $\vec{u}_r$ and $\vec{u}_\theta$.

10. Express the acceleration in terms of the local coordinates $\vec{u}_r$ and $\vec{u}_\theta$.

11. Compare and contrast the proof of Theorems 1.1 and 1.2 given in Section 1.5 with Newton's original proof. Which proof do you like better? Why?

12. The constant ε in Equation (1.18) is called the *eccentricity.* What happens if ε is larger than 1? equal to 1? equal to 0? less than 0?

13. If $\varepsilon = 1$ in Equation (1.18), then there is a value of θ, $\theta = \pi$, for which r is not defined. If $\varepsilon = -1$, then r is not defined when $\theta = 0$. If $|\varepsilon|$ is larger that one, then there is an interval of values for θ over which r is not defined. Find this interval in terms of ε and explain its relationship to the path $r(1 + \varepsilon\cos\theta) = c$ when $|\varepsilon| > 1$.

14. Prove that the constant c in Equation (1.18) is half of the *latus rectum* L of the ellipse.

15. Comparing Equations (1.5) and (1.20) we see that $|RS|$ is the change in velocity per unit time, which corresponds to $|\vec{a}(t)|$; $2A$ is twice the area swept out per unit time, which corresponds to k; $|PF_1|$ is the distance from the sun, which corresponds to r; but L is twice c. Explain this discrepancy.

16. Find the acceleration in terms of distance from the origin of a particle moving along an ellipse with its center at the origin (instead of having one focus at the origin) and sweeping out equal area in equal time.

17. Find the acceleration in terms of distance from the origin of a particle moving along the logarithmic spiral

$$r = e^{-c\theta},$$

where c is an arbitrary constant, given that the particle sweeps out equal area in equal time.

18. Given a particle that sweeps out an equal area in equal time and whose path is given by $r = f(\theta)$, show that the acceleration is given by

$$\vec{a} = \frac{k^2}{r^3}\left[\frac{f''(\theta)}{f(\theta)} - 2\left(\frac{f'(\theta)}{f(\theta)}\right)^2 - 1\right]\vec{u}_r.$$

19. Find the acceleration in terms of distance from the origin of a particle with constant angular velocity,

$$\frac{d\theta}{dt} = k,$$

which follows an elliptical orbit with the sun at one focus,

$$r(1 + \varepsilon\cos\theta) = c.$$

20. If we use complex coordinates to represent the points in the plane of the orbit, then we have the correspondence

$$\vec{u}_r = (\cos\theta, \sin\theta) \longleftrightarrow e^{i\theta} = \cos\theta + i\sin\theta.$$

Show that $\vec{u}_\theta$ corresponds to $ie^{i\theta}$.

21. Show that if

$$\vec{r} = re^{i\theta}, \tag{1.26}$$

where r and θ are functions of time t, then

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt} e^{i\theta} + r \frac{d\theta}{dt} i e^{i\theta} \tag{1.27}$$

and

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 \right) e^{i\theta} + \frac{1}{r} \frac{d}{dt} \left(r^2 \frac{d\theta}{dt} \right) i e^{i\theta}. \tag{1.28}$$