Some Elliptical Derivations

The first derivation is to show that the equation of an ellipse centred at the origin is given by an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If you want to trust this, that's ok. I thought it was worth seeing why it was true. If you want to accept it as having been true for 5+ years of your life, that's ok.

Here's some things that are worth knowing about an ellipse - I will draw a picture of this in the video.

An ellipse begins with two points - the foci. This is rather quite important for celestial mechanics, as the object being orbited around is located at one of the foci. The set of points on an ellipse is the set of all points such that the sum of the two distances to the two foci is constant. The maximal distance from the centre of an ellipse is the semi-major axis. The minimal distance from the centre of an ellipse is the semi-minor axis. They are like the two different radii of the ellipse. It is common to choose coordinates so that the semi-major axis is parallel to the x-axis and he semi-minor axes is parallel to the y-axis. In this case, $a =$ the semi-major axis, and $b =$ the semi-major axis.

I will start with assuming the centre is at the origin and the semi-major axis is parallel to the x-axis, so the foci are on the x-axis and equidistant to the origin, let's say at the points $(f, 0)$ and $(-f, 0)$. Furthermore let's say that the sum of the distances between a point on the ellipse to the two foci is equal to d. From a quick visual analysis (which I will show in the video), we can see from a point on the y-axis that $b^2 + f^2 = (\frac{d}{2})^2$, and $a = \frac{d}{2}$, hence $b^2 + f^2 = a^2$. Here remember that f is the distance of either focus from the centre of the ellipse. It's also worth noting that ellipses have an eccentricity, $\epsilon = \frac{\sqrt{a^2-b^2}}{a}$ which is always between 0 and 1. If $a = b$, $\epsilon = 0$, which gives a circle. If $b = 0$, $\epsilon = 1$, which gives a degenerate flat ellipse.

All of the above is important. From this we can derive the equation of an ellipse. The result is important, the steps to get there might not be. Away we go:

Suppose the point on the ellipse is (x, y). We know the sum of the two distances from this point to the foci is d. So, the locus of points on the ellipse are those satisfying the equation

$$\sqrt{(x-f)^2 + y^2} + \sqrt{(x+f)^2 + y^2} = d$$

Now we move on square root to the right:

$$\sqrt{(x-f)^2 + y^2} = d - \sqrt{(x+f)^2 + y^2}$$

And square both sides:

$$(x-f)^2 + y^2 = d^2 + (x+f)^2 + y^2 - 2d\sqrt{(x+f)^2 + y^2}$$

Expand squares:

$$x^2 - 2xf + f^2 + y^2 = d^2 + x^2 + 2xf + f^2 + y^2 - 2d\sqrt{(x+f)^2 + y^2}$$

Then cancel and move everything but the square root to the right and the square root to the left to get

$$2d\sqrt{(x+f)^2 + y^2} = d^2 + 4xf$$

Again square both sides and expand to get

$$4d^2x^2 + 8d^2xf + 4d^2f^2 + 4d^2y^2 = d^4 + 8d^2xf + 16x^2f^2$$

Cancel and switch sides for $16x^2f^2$ and $4d^2f^2$, yielding

$$4d^2x^2 - 16x^2f^2 + 4d^2y^2 = d^4 - 4d^2f^2$$

Factor d^2 out of the RHS and $4x^2$ out of the first two terms on the left.

$$4x^2(d^2 - 4f^2) + 4d^2y^2 = d^2(d^2 - 4f^2)$$

Remember from above that $b^2 + f^2 = (\frac{d}{2})^2$ This tells us that $4b^2 = d^2 - 4f^2$. This isn't necessary, but surely makes it easier to write giving us

$$4x^2(4b^2) + 4d^2y^2 = d^2 4b^2$$

Factor our 4s, and then divide by b^2d^2 to get

$$\frac{4x^2}{d^2} + \frac{y^2}{b^2} = 1$$

Remember one more thing from above, that $a = \frac{d}{2}$, and this produces

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

as I hope you have long known. Ok, so that's where that comes from.

The next thing that Bressoud talks about is a polar form for an ellipse with one focus at the origin (as we would have for an object orbiting according to Kepler's elliptical orbits). He gives the polar form first and then proves it's the same as the rectangular form. This makes the algebra easier, but seems fake. Where did the polar form come from? Here's the work in the opposite direction. I'm typing this because it's tedious and this allows us to read it more quickly. Again, the argument in Bressoud shows that it is true, but not where it comes from.

If we have an ellipse with a focus at the origin, say the right focus, then we shift the equation to the left by f to get

$$1 = \frac{(x+f)^2}{a^2} + \frac{y^2}{b^2}$$

Taking a couple steps at once, clear fractions and expand to get:

$$a^2b^2 = x^2b^2 + 2xfb^2 + b^2f^2 + a^2y^2$$

Move two terms to the left:

$$a^2b^2 - 2xfb^2 - b^2f^2 = x^2b^2 + a^2y^2$$

Factor b^2 out of the left and divide both sides by a^2:

$$\frac{b^2(a^2 - 2xf - f^2)}{a^2} = \frac{x^2b^2}{a^2} + y^2$$

As an insight, I am trying to work toward polar coordinates, so I am working to get $x^2+y^2 = r^2$ on the right. I now have y^2 . To get x^2, we subtract the rest over to the other side. To see this more clearly, I introduce 1-1 in to the coefficient:

$$\frac{b^2(a^2 - 2xf - f^2)}{a^2} = x^2(1 - 1 + \frac{b^2}{a^2}) + y^2$$

Now move the extra x^2s to the left to get

$$x^2(1 - \frac{b^2}{a^2}) + \frac{b^2(a^2 - 2xf - f^2)}{a^2} = x^2 + y^2$$

Notice that $(1 - \frac{b^2}{a^2}) = \frac{a^2 - b^2}{a^2}$. Recall for notational convenience that $b^2 + f^2 = a^2$, so $a^2 - b^2 = f^2$. Also, $a^2 - f^2 = b^2$. Making these two substitutions and expanding gives

$$\frac{x^2 f^2}{a^2} - \frac{2xfb^2}{a^2} + \frac{b^4}{a^2} = x^2 + y^2$$

As a happy coincidence the left side is a perfect square, gathering that together and admitting that the right side is r^2

$$(\frac{xf - b^2}{a})^2 = r^2$$

We will next take the square root of both sides. Either choice of $\pm$ will make this equation true, so I will change the sign on the left to make it work out more nicely. From this, and recalling that $x = r\cos\theta$ I get

$$\frac{b^2}{a} - \frac{fr\cos\theta}{a} = r$$

To make this simpler and to fit with the book's notation, let $c = \frac{b^2}{a}$. Notice that a and b are axes, so if we have an actual ellipse, both are positive, hence $c > 0$. Also let's use $\epsilon = \frac{f}{a}$ Please note this is the same as above if f is positive. If f is negative it means we started with the left focus at the origin instead of the right focus at the origin. Notice also that $|\epsilon| < 1$ if we have an actual ellipse (not a line segment). Making these two substitutions produces

$$c - \epsilon r\cos\theta = r$$

We can move all rs to the same side to get the equation in the book

$$r + \epsilon r\cos\theta = r(1 + \epsilon\cos\theta) = c$$

Or if you want something you can graph using a calculator or software we can solve for r to get $r = c/(1 + \epsilon\cos\theta)$. Notice for a little more, as noticed in the book, that the semi-major axis is

$$\frac{c}{1 - \epsilon^2} = \frac{\frac{b^2}{a}}{\frac{a^2 - (a^2 - b^2)}{a^2}} = a$$

and the semi-minor axis is

$$\frac{c}{\sqrt{1 - \epsilon^2}} = \frac{\frac{b^2}{a}}{\frac{b}{a}} = b$$

And that's how you get these answers without knowing them to start. Please note: these typed notes are not important, they are typed so I can go through them quickly. The part after it is important and I will follow the text and present that on the board.