Problem Set 3 Exemplars

Here are my comments:

4.5 5 was mostly computational. Notice that you don't feel as if you are integrating because the form is constant.

4.5.6 this requires *any* constant flow. To do them all, you must use an arbitrary velocity vector, such as (v_1, v_2, v_3). Still it is not very demanding.

4.5 7 this problem requires use of pullbacks (although it can be done without, but not to complete the stated problem). I am convinced that the best way to do this is to parametrise the *entire* tetrahedron in 3-dimensions (as demonstrated on pp. 99-100) and then to pull back to the faces one-by-one. The tricky part there is dealing with the face that doesn't include the origin and does not have area 1/2, *but* that face projects to triangles that *do* include the origin and have area 1/2. As in 6, this one face cancels with all of the other three together. Each of the projected pullbacks are identical to the other faces, but with opposite orientation, thus canceling. No one completely followed my suggestions in this direction. The examplar is a way to fight through all the algebra to do this with individual parametrisations of each face. As one more detail, to get the 3-d parametrisation approach to work you will need to divide the exceptional face into three integrals, one for each of the associated basic two-forms.

4.8 6 should have been basic practice with exterior algebra. There is nothing more here than expanding and simplifying the multiplication of differential forms.

4.8 13 The idea, as demonstrated in the exemplar, is to find higher dimensional volumes by adding up cross sections in lower dimensions (you did this from dimension 2 to 3 in Calc I or II depending on where and when you took it). I do not expect a careful proof by induction. Induction loosely is the mathematical way of saying "and so on". There are no computations here (the integral is given to you), but the most important part is explaining why the integral computes volumes.

4.8 17 This is a standard Calc III line integral. Really no one seemed to have a problem with it.

4.5

5. $v = (3, 0, -1)$ $\qquad \frac{1}{2}(\vec{b}-\vec{a}) \times (\vec{c}-\vec{a})$

a. $\frac{1}{2}\left((5-(-2)), (3-1), (-1-0)\right) \times \left((8-(-2)), (-5-1), (2-0)\right)$

$= \frac{1}{2}(7, 2, -1) \times (10, -6, 2)$

$= \frac{1}{2}\left((2\cdot 2)-(-1\cdot -6), (-1\cdot 10)-(7\cdot 2), (7\cdot -6)-(2\cdot 10)\right)$

$= \frac{1}{2}(-2, -24, -62) = (-1, -12, -31)$

$\int_T dx\,dy = -31, \int_T dz\,dx = -12, \int_T dy\,dz = -1$

$v_1 \int_T dy\,dz + v_2 \int dz\,dx + v_3 \int dx\,dy = (3)(-1) + (0)(-12) + (-1)(-31)$

$= (31) - (3) = 28$

b. $\frac{1}{2}(-3, 8, -3) \times (-10, 6, -2) = \frac{1}{2}(2, 24, 62) = (1, 12, 31)$

$v_1 \int_T dy\,dz + v_2 \int_T dz\,dx + v_3 \int_T dx\,dy = (3)(1) + (0)(12) + (-1)(31) = (3) - 31$

$= -28$

c. $\frac{1}{2}(1, 1, 2) \times (2, -1, -3) = \frac{1}{2}(-1, 7, -3) = \left(-\frac{1}{2}, \frac{7}{2}, -\frac{3}{2}\right)$

$v_1 \int_T dy\,dz + v_2 \int_T dz\,dx + v_3 \int_T dx\,dy = (3)\left(-\frac{1}{2}\right) + (0)\left(-\frac{7}{2}\right) + (-1)\left(-\frac{3}{2}\right) = \frac{3}{2} - \frac{3}{2} = 0$

d. $\frac{1}{2}(0, 1, 0) \times (1, 0, 0) = \frac{1}{2}(0, 0, -1) = \left(0, 0, -\frac{1}{2}\right)$

$v_1 \int_T dy\,dz + v_2 \int_T dz\,dx + v_3 \int dx\,dy = 0 + 0 + (-1)\left(-\frac{1}{2}\right) = \boxed{\frac{1}{2}}$

$$\vec{v} = (3, 0, -1) \qquad T = [\vec{a}, \vec{b}, \vec{c}]$$

e. $\vec{a} = (1,0,0)$ $\vec{b} = (0,1,0)$ $\vec{c} = (0,0,1)$ $T = [\vec{a}, \vec{b}, \vec{c}]$

$$\tfrac{1}{2}(-1,1,0) \times (-1,0,1) = \tfrac{1}{2}\big((1\cdot 1)-(1\cdot 0), (0\cdot -1)-(-1\cdot 1), (-1\cdot 0)-(1\cdot -1)\big)$$

$$= \tfrac{1}{2}(1,1,1) = \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$v_1\int_T dy\,dz + v_2\int_T dz\,dx + v_3\int_T dx\,dy = (3)\left(\tfrac{1}{2}\right) + 0\left(\tfrac{1}{2}\right) + (-1)\left(\tfrac{1}{2}\right) = \tfrac{3}{2} - \tfrac{1}{2} = \tfrac{2}{2} = \boxed{1}$$

f. $\vec{a} = (0,0,0)$ $\vec{b} = (0,0,1)$ $\vec{c} = (0,1,0)$

$$\tfrac{1}{2}(0,0,1) \times (0,1,0) = \tfrac{1}{2}(-1,0,0) = \left(-\tfrac{1}{2}, 0, 0\right)$$

$$v_1\int_T dy\,dz + v_2\int_T dz\,dx + v_3\int_T dx\,dy = (3)\left(-\tfrac{1}{2}\right) + 0 + 0 = \boxed{-\tfrac{3}{2}}$$

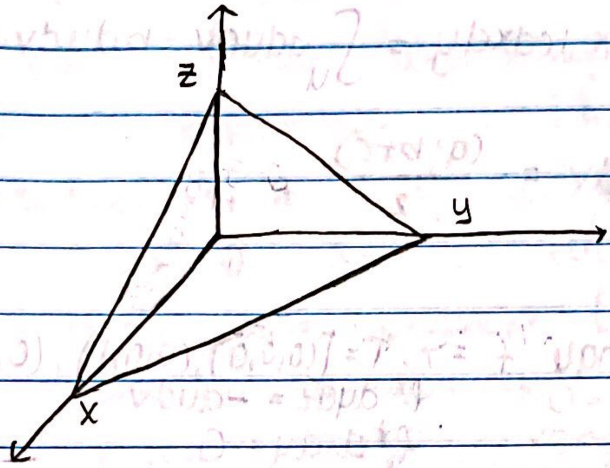
g. $\vec{a} = (0,0,0)$, $\vec{b} = (1,0,0)$, $\vec{c} = (0,0,1)$

$$\tfrac{1}{2}(1,0,0) \times (0,0,1) = \tfrac{1}{2}(0,-1,0) = \left(0, -\tfrac{1}{2}, 0\right)$$

$$v_1\int_T dy\,dz + v_2\int_T dz\,dx + v_3\int_T dx\,dy = 0 + 0 + 0 = \boxed{0}$$

#6 Prove that for any constant flow, the sum of the rates at which the fluid crosses each of these faces is zero.

The faces of a tetrahedron (Figure 4.6):



Looking at parts d through g in #5, the sum of the of the rates is zero.

$$\frac{1}{2} + 1 + -\frac{3}{2} + 0 = 0$$

$$d + e + f + g = 0$$

To prove for any constant flow, let constant velocity $\vec{v} = (a, b, c)$ crosses the triangles d through g.

$$\vec{v} = (a, b, c) \Rightarrow a\,dy\,dz + b\,dz\,dx + c\,dx\,dy$$

looking @ triangle d $\Rightarrow T[(0,0,0), (0,1,0), (1,0,0)]$

$$\varphi\begin{pmatrix}u\\v\end{pmatrix} = \begin{array}{cc}\begin{pmatrix}v\\u\\0\end{pmatrix} & \begin{array}{c}x\\y\\z\end{array}\end{array}$$

$\varphi^* dx = dv$ $\quad \varphi^* dy\,dz = 0$

$\varphi^* dy = du$ $\quad \varphi^* dx\,dy = dv\,du = -du\,dv$

$\varphi^* dz = 0$ $\quad \varphi^* dz\,dx = 0$

$$\int_T a\,dy\,dz + b\,dz\,dx + c\,dx\,dy = \int_U -c\,du\,dv = \frac{-c}{2}$$

looking at triangle e => $T = [(1,0,0),(0,1,0),(0,0,1)]$

$$\varphi\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}1-u-v\\u\\v\end{pmatrix}\begin{matrix}x\\y\\z\end{matrix} \qquad \begin{matrix}\varphi^* dx = -du - dv \\ \varphi^* dy = du \\ \varphi^* dz = dv\end{matrix} \qquad \begin{matrix}\varphi^* dydz = dudv \\ \varphi^* dxdy = -dvdu = dudv \\ \varphi^* dzdx = -dvdu = dudv\end{matrix}$$

$$\int_T \cancel{adydz + bdzdx + cdxdy} = \int_u \cancel{adudv + bdudv + cdudv}$$

$$= \int_u (a+b+c)dudv = \frac{(a+b+c)}{2}$$

looking at triangle f => $T = [(0,0,0),(0,0,1),(0,1,0)]$

$$\varphi\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}0\\v\\u\end{pmatrix}\begin{matrix}x\\y\\z\end{matrix} \qquad \begin{matrix}\varphi^* dx = 0 \\ \cancel{\varphi^* dy = dv} \\ \varphi^* dz = du\end{matrix} \qquad \begin{matrix}\varphi^* dydz = -dudv \\ \cancel{\varphi^* dxdy = 0} \\ \varphi^* dzdx = 0\end{matrix}$$

$$\int_u -adudv = \frac{-a}{2}$$

looking @ triangle g => $T = [(0,0,0),(1,0,0),(0,0,1)]$

$$\varphi\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}u\\0\\v\end{pmatrix}\begin{matrix}x\\y\\z\end{matrix} \qquad \begin{matrix}\varphi^* dx = du \\ \varphi^* dy = 0 \\ \varphi^* dz = dv\end{matrix} \qquad \begin{matrix}\varphi^* dydz = 0 \\ \varphi^* dxdy = 0 \\ \varphi^* dzdx = dvdu = -dudv\end{matrix}$$

$$\int_u -bdudv = \frac{-b}{2}$$

Then the addition of d through g

$$\frac{(a+b+c)}{2} + \frac{-a}{2} + \frac{-b}{2} + \frac{-c}{2} = 0$$

Let the flow have velocity **v** = (k_1, k_2, k_3) and work done over a triangle T be

$$\int_T k_1 dydz + k_2 dzdx + k_3 dxdy$$

Let the four faces on the tetrahedron be the triangles D = [(a_1, a_2, a_3), (c_1, c_2, c_3), (b_1, b_2, b_3)], E = [(b_1, b_2, b_3), (c_1, c_2, c_3), (d_1, d_2, d_3)], F = [(a_1, a_2, a_3), (d_1, d_2, d_3), (c_1, c_2, c_3)], and G = [(a_1, a_2, a_3), (b_1, b_2, b_3), (d_1, d_2, d_3)]. Additionally, U is a triangle such that U = [(0,0), (1,0), (0,1)].

For D, the amount of work done is

$$\int_D k_1 dydz + k_2 dzdx + k_3 dxdy$$

We begin with the parametrization:

$$\varphi \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a_1 + (c_1 - a_1)u + (b_1 - a_1)v \\ a_2 + (c_2 - a_2)u + (b_2 - a_2)v \\ a_3 + (c_3 - a_3)u + (b_3 - a_3)v \end{pmatrix}$$

From this we can do the following pullbacks:

$$\varphi^*(dx) = (c_1 - a_1)du + (b_1 - a_1)dv$$

$$\varphi^*(dy) = (c_2 - a_2)du + (b_2 - a_2)dv$$

$$\varphi^*(dz) = (c_3 - a_3)du + (b_3 - a_3)dv$$

$$\varphi^*(dydz) = \big((c_2 - a_2)(b_3 - a_3) - (b_2 - a_2)(c_3 - a_3)\big)dudv$$

$$\varphi^*(dzdx) = \big((c_3 - a_3)(b_1 - a_1) - (b_3 - a_3)(c_1 - a_1)\big)dudv$$

$$\varphi^*(dxdy) = \big((c_1 - a_1)(b_2 - a_2) - (b_1 - a_1)(c_2 - a_2)\big)dudv$$

Therefore we can calculate the work done:

$$\int_D k_1 dydz + k_2 dzdx + k_3 dxdy$$

$$= \int_U k_1\big((c_2 - a_2)(b_3 - a_3) - (b_2 - a_2)(c_3 - a_3)\big)$$
$$+ k_2\big((c_3 - a_3)(b_1 - a_1) - (b_3 - a_3)(c_1 - a_1)\big)$$
$$+ k_3\big((c_1 - a_1)(b_2 - a_2) - (b_1 - a_1)(c_2 - a_2)\big)\, dudv$$

$$= \Big(k_1\big((c_2 - a_2)(b_3 - a_3) - (b_2 - a_2)(c_3 - a_3)\big)$$
$$+ k_2\big((c_3 - a_3)(b_1 - a_1) - (b_3 - a_3)(c_1 - a_1)\big)$$
$$+ k_3\big((c_1 - a_1)(b_2 - a_2) - (b_1 - a_1)(c_2 - a_2)\big)\Big)/2$$

$$= \big(k_1((c_2b_3 - c_2a_3 - a_2b_3 + a_2a_3) - (b_2c_3 - b_2a_3 - a_2c_3 + a_2a_3))$$
$$+ k_2((c_3b_1 - c_3a_1 - a_3b_1 + a_3a_1) - (b_3c_1 - b_3a_1 - a_3c_1 + a_3a_1))$$
$$+ k_3((c_1b_2 - c_1a_2 - a_1b_2 + a_1a_2) - (b_1c_2 - b_1a_2 - a_1c_2 + a_1a_2))\big)/2$$

$$= (k_1(c_2b_3 - c_2a_3 - a_2b_3 - b_2c_3 + b_2a_3 + a_2c_3) + k_2(c_3b_1 - c_3a_1 - a_3b_1 - b_3c_1 + b_3a_1$$
$$+ a_3c_1) + k_3(c_1b_2 - c_1a_2 - a_1b_2 - b_1c_2 + b_1a_2 + a_1c_2))/2$$

The same will now be done for E:

$$\varphi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} b_1 + (c_1 - b_1)u + (d_1 - b_1)v \\ b_2 + (c_2 - b_2)u + (d_2 - b_2)v \\ b_3 + (c_3 - b_3)u + (d_3 - b_3)v \end{pmatrix}$$

$$\varphi^*(dx) = (c_1 - b_1)du + (d_1 - b_1)dv$$

$$\varphi^*(dy) = (c_2 - b_2)du + (d_2 - b_2)dv$$

$$\varphi^*(dz) = (c_3 - b_3)du + (d_3 - b_3)dv$$

$$\varphi^*(dydz) = \big((c_2 - b_2)(d_3 - b_3) - (d_2 - b_2)(c_3 - b_3)\big)dudv$$

$$\varphi^*(dzdx) = \big((c_3 - b_3)(d_1 - b_1) - (d_3 - b_3)(c_1 - b_1)\big)dudv$$

$$\varphi^*(dxdy) = \big((c_1 - b_1)(d_2 - b_2) - (d_1 - b_1)(c_2 - b_2)\big)dudv$$

$$\int_E k_1 dydz + k_2 dzdx + k_3 dxdy$$

$$= \int_U k_1\big((c_2 - b_2)(d_3 - b_3) - (d_2 - b_2)(c_3 - b_3)\big)$$

$$+ k_2\big((c_3 - b_3)(d_1 - b_1) - (d_3 - b_3)(c_1 - b_1)\big)$$

$$+ k_3\big((c_1 - b_1)(d_2 - b_2) - (d_1 - b_1)(c_2 - b_2)\big)\, dudv$$

$$= (k_1(c_2d_3 - c_2b_3 - b_2d_3 - d_2c_3 + d_2b_3 + b_2c_3) + k_2(c_3d_1 - c_3b_1 - b_3d_1 - d_3c_1 + d_3b_1$$

$$+ b_3c_1) + k_3(c_1d_2 - c_1b_2 - b_1d_2 - d_1c_2 + d_1b_2 + b_1c_2))/2$$

Now for F:

$$\varphi\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}a_1 + (d_1 - a_1)u + (c_1 - a_1)v\\ a_2 + (d_2 - a_2)u + (c_2 - a_2)v\\ a_3 + (d_3 - a_3)u + (c_3 - a_3)v\end{pmatrix}$$

$$\varphi^*(dx) = (d_1 - a_1)du + (c_1 - a_1)dv$$

$$\varphi^*(dy) = (d_2 - a_2)du + (c_2 - a_2)dv$$

$$\varphi^*(dz) = (d_3 - a_3)du + (c_3 - a_3)dv$$

$$\varphi^*(dydz) = \big((d_2 - a_2)(c_3 - a_3) - (c_2 - a_2)(d_3 - a_3)\big)dudv$$

$$\varphi^*(dzdx) = \big((d_3 - a_3)(c_1 - a_1) - (c_3 - a_3)(d_1 - a_1)\big)dudv$$

$$\varphi^*(dxdy) = \big((d_1 - a_1)(c_2 - a_2) - (c_1 - a_1)(d_2 - a_2)\big)dudv$$

$$\int_F k_1 dydz + k_2 dzdx + k_3 dxdy$$

$$= \int_U k_1\big((d_2 - a_2)(c_3 - a_3) - (c_2 - a_2)(d_3 - a_3)\big)$$

$$+ k_2\big((d_3 - a_3)(c_1 - a_1) - (c_3 - a_3)(d_1 - a_1)\big)$$

$$+ k_3\big((d_1 - a_1)(c_2 - a_2) - (c_1 - a_1)(d_2 - a_2)\big)\, dudv$$

$$= (k_1(d_2c_3 - d_2a_3 - a_2c_3 - c_2d_3 + c_2a_3 + a_2d_3) + k_2(d_3c_1 - d_3a_1 - a_3c_1 - c_3d_1 + c_3a_1$$

$$+ a_3d_1) + k_3(d_1c_2 - d_1a_2 - a_1c_2 - c_1d_2 + c_1a_2 + a_1d_2))/2$$

Now G:

$$\varphi\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a_1 + (b_1 - a_1)u + (d_1 - a_1)v \\ a_2 + (b_2 - a_2)u + (d_2 - a_2)v \\ a_3 + (b_3 - a_3)u + (d_3 - a_3)v \end{pmatrix}$$

$$\varphi^*(dx) = (b_1 - a_1)du + (d_1 - a_1)dv$$

$$\varphi^*(dy) = (b_2 - a_2)du + (d_2 - a_2)dv$$

$$\varphi^*(dz) = (b_3 - a_3)du + (d_3 - a_3)dv$$

$$\varphi^*(dydz) = \big((b_2 - a_2)(d_3 - a_3) - (d_2 - a_2)(b_3 - a_3)\big)dudv$$

$$\varphi^*(dzdx) = \big((b_3 - a_3)(d_1 - a_1) - (d_3 - a_3)(b_1 - a_1)\big)dudv$$

$$\varphi^*(dxdy) = \big((b_1 - a_1)(d_2 - a_2) - (d_1 - a_1)(b_2 - a_2)\big)dudv$$

$$\int_F k_1 dydz + k_2 dzdx + k_3 dxdy$$

$$= \int_U k_1\big((b_2 - a_2)(d_3 - a_3) - (d_2 - a_2)(b_3 - a_3)\big)$$
$$+ k_2\big((b_3 - a_3)(d_1 - a_1) - (d_3 - a_3)(b_1 - a_1)\big)$$
$$+ k_3\big((b_1 - a_1)(d_2 - a_2) - (d_1 - a_1)(b_2 - a_2)\big)\, dudv$$

$$= (k_1(b_2d_3 - b_2a_3 - a_2d_3 - d_2b_3 + d_2a_3 + a_2b_3) + k_2(b_3d_1 - b_3a_1 - a_3d_1 - d_3b_1 + d_3a_1$$
$$+ a_3b_1) + k_3(b_1d_2 - b_1a_2 - a_1d_2 - d_1b_2 + d_1a_2 + a_1b_2))/2$$

Now we will add together all of these rates:

$$\int_D k_1 dydz + k_2 dzdx + k_3 dxdy + \int_E k_1 dydz + k_2 dzdx + k_3 dxdy$$
$$+ \int_F k_1 dydz + k_2 dzdx + k_3 dxdy + \int_G k_1 dydz + k_2 dzdx + k_3 dxdy$$

$$= (k_1(c_2b_3 - c_2a_3 - a_2b_3 - b_2c_3 + b_2a_3 + a_2c_3) + k_2(c_3b_1 - c_3a_1 - a_3b_1 - b_3c_1 + b_3a_1$$
$$+ a_3c_1) + k_3(c_1b_2 - c_1a_2 - a_1b_2 - b_1c_2 + b_1a_2 + a_1c_2))/2$$
$$+ (k_1(c_2d_3 - c_2b_3 - b_2d_3 - d_2c_3 + d_2b_3 + b_2c_3) + k_2(c_3d_1 - c_3b_1 - b_3d_1$$
$$- d_3c_1 + d_3b_1 + b_3c_1) + k_3(c_1d_2 - c_1b_2 - b_1d_2 - d_1c_2 + d_1b_2 + b_1c_2))/2$$
$$+ (k_1(d_2c_3 - d_2a_3 - a_2c_3 - c_2d_3 + c_2a_3 + a_2d_3) + k_2(d_3c_1 - d_3a_1 - a_3c_1$$
$$- c_3d_1 + c_3a_1 + a_3d_1) + k_3(d_1c_2 - d_1a_2 - a_1c_2 - c_1d_2 + c_1a_2 + a_1d_2))/2$$
$$+ (k_1(b_2d_3 - b_2a_3 - a_2d_3 - d_2b_3 + d_2a_3 + a_2b_3) + k_2(b_3d_1 - b_3a_1 - a_3d_1$$
$$- d_3b_1 + d_3a_1 + a_3b_1) + k_3(b_1d_2 - b_1a_2 - a_1d_2 - d_1b_2 + d_1a_2 + a_1b_2))/2$$

$$
\begin{aligned}
= (k_1(&c_2b_3 - c_2b_3 + c_2a_3 - c_2a_3 + a_2b_3 - a_2b_3 + b_2c_3 - b_2c_3 + b_2a_3 - b_2a_3 + a_2c_3 - a_2c_3 \\
&+ c_2d_3 - c_2d_3 + b_2d_3 - b_2d_3 + d_2c_3 - d_2c_3 + d_2b_3 - d_2b_3 + d_2a_3 - d_2a_3 \\
&+ a_2d_3 - a_2d_3) \\
&+ k_2(c_3b_1 - c_3b_1 + c_3a_1 - c_3a_1 + a_3b_1 - a_3b_1 + b_3c_1 - b_3c_1 + b_3a_1 - b_3a_1 \\
&+ a_3c_1 - a_3c_1 + c_3d_1 - c_3d_1 + b_3d_1 - b_3d_1 + d_3c_1 - d_3c_1 + d_3b_1 - d_3b_1 \\
&+ d_3a_1 - d_3a_1 + a_3d_1 - a_3d_1) \\
&+ k_3(c_1b_2 - c_1b_2 + c_1a_2 - c_1a_2 + a_1b_2 - a_1b_2 + b_1c_2 - b_1c_2 + b_1a_2 - b_1a_2 \\
&+ a_1c_2 - a_1c_2 + c_1d_2 - c_1d_2 + b_1d_2 - b_1d_2 + d_1c_2 - d_1c_2 + d_1b_2 - d_1b_2 \\
&+ d_1a_2 - d_1a_2 + a_1d_2 - a_1d_2))/2
\end{aligned}
$$

It is clear that because every term in the above expression is added to its inverse, the total sum is equal to 0. Therefore, the total net rate at which any constant flow crosses the surface of any tetrahedron is zero.

4.8.6

Consider
$$\begin{cases} y_1 = 3 - 2x_1 + 5x_2 - 3x_3 \\ y_2 = x_1 - 3x_3 + 2x_4 \\ y_3 = -1 + 5x_1 - x_2 + 4x_4 \\ y_4 = 6 + 2x_2 - x_4 \end{cases}$$

$$dy_1 = -2\,dx_1 + 5\,dx_2 - 3\,dx_3$$
$$dy_2 = dx_1 - 3\,dx_3 + 2\,dx_4$$
$$dy_3 = 5\,dx_1 - dx_2 + 4\,dx_4$$
$$dy_4 = 2\,dx_2 - dx_4$$

a. $dy_1\, dy_3$

$$= (-2dx_1 + 5dx_2 - 3dx_3)(5dx_1 - dx_2 + 4dx_4)$$
$$= 10(0) + 2\,dx_1dx_2 - 8dx_1dx_4 + 25dx_2dx_1 - 5(0) + 20\,dx_2dx_4 - 15dx_3dx_1 + 3dx_3dx_2 - 12dx_3dx_4$$
$$= 2dx_1dx_2 - 8\,dx_1dx_4 + 25dx_2dx_1 + 20\,dx_2dx_4 - 15dx_3dx_1 + 3dx_3dx_2 - 12\,dx_3dx_4$$
$$= -2\,dx_2dx_1 - 8dx_1dx_4 + 25dx_2dx_1 + 20\,dx_2dx_4 - 15dx_3dx_1 + 3dx_3dx_2 - 12\,dx_3dx_4$$
$$= 23\,dx_2dx_1 - 8\,dx_1dx_4 + 20\,dx_2dx_4 - 15dx_3dx_1 + 3\,dx_3dx_2 - 12dx_3dx_4$$
$$= -23dx_1dx_2 + 15\,dx_3dx_1 - 8\,dx_1dx_4 - 3dx_2dx_3 + 20dx_2dx_4 - 12\,dx_3dx_4$$

b $dy_1\, dy_4$

$$= (-2dx_1 + 5dx_2 - 3dx_3)(2dx_2 - dx_4)$$
$$= -4\,dx_1dx_2 + 2\,dx_1dx_4 + 10(0) - 5dx_2\,dx_4 - 6dx_3dx_2 + 3\,dx_3dx_4$$
$$= -4\,dx_1dx_2 + 2dx_1dx_4 + 6\,dx_2dx_3 - 5dx_2dx_4 + 3dx_3dx_4$$

4.8.6 cont.

c. $dy_2 dy_3 dy_4$

$(dx_1 - 3dx_3 + 2dx_4)(5dx_1 - dx_2 + 4dx_4)(2dx_2 - dx_4)$

$= (5(0) - dx_1dx_2 + 4dx_1dx_4 - 15dx_3dx_1 - 3dx_3dx_2 - 12dx_3dx_4 + 10dx_4dx_1 - 2dx_4dx_2 + 8(0))$
$(2dx_2 - dx_4)$

$= (-dx_1dx_2 + 15dx_1dx_3 - 6dx_1dx_4 + 3dx_2dx_3 + 2dx_2dx_4 - 12dx_3dx_4)(2dx_2 - dx_4)$

$= -dx_1(0) + 30dx_1dx_3dx_2 - 12dx_1dx_4dx_2 + 6dx_3(0) + 4dx_4(0) - 24dx_3dx_4dx_2$
$+ dx_1dx_2dx_4 - 15dx_1dx_3dx_4 + 6dx_1(0) - 3dx_3dx_2dx_4 + 2dx_2(0) + 12dx_3(0)$

$= 30dx_1dx_3dx_2 - 12dx_1dx_4dx_2 - 24dx_3dx_4dx_2 + dx_1dx_2dx_4 - 15dx_1dx_3dx_4$
$+ 3dx_2dx_3dx_4$

$= -30dx_1dx_2dx_3 + 13dx_1dx_2dx_4 - 15dx_1dx_3dx_4 + 24dx_3dx_2dx_4$
$+ 3dx_2dx_3dx_4$

$= -30dx_1dx_2dx_3 + 13dx_1dx_2dx_4 - 15dx_1dx_3dx_4 + (-24 + 3)dx_2dx_3dx_4$

$= -30dx_1dx_2dx_3 + 13dx_1dx_2dx_4 - 15dx_1dx_3dx_4 - 21dx_2dx_3dx_4$

d. $dy_1 dy_2 dy_3 dy_4$

$= (-2dx_1 + 5dx_2 - 3dx_3)(-30dx_1dx_2dx_3 + 13dx_1dx_2dx_4 - 15dx_1dx_3dx_4 - 21dx_2dx_3dx_4)$

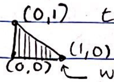
$= 42dx_1dx_2dx_3dx_4 - 75dx_2dx_1dx_3dx_4 - 39dx_3dx_1dx_2dx_4$

$= 42dx_1dx_2dx_3dx_4 + 75dx_1dx_2dx_3dx_4 - 39dx_1dx_2dx_3dx_4$

$\boxed{= 78\, dx_1dx_2dx_3dx_4}$

13) Using induction and the fact that $\int_0^1 \frac{t^{k-1}}{(k-1)} dt = \frac{1}{k!}$, justify the statement that the hypervolume of a fundamental k-simplex is $1/k!$.

So, let test hypervolume of 2 simplex, 3 simplex, 4 simplex

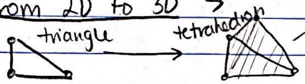
from 1D to 2D 2 simplex • →  t = height triangle

we break into slices for integration

$\int_0^1 t\,dt \rightarrow 1D \rightarrow \int_0^1 \frac{t^{2-1}}{(2-1)!} = \frac{1}{2!} = \frac{1}{2}$ for a 2-simplex...

checks out with $\frac{1}{k!}$! ✓

from 2D to 3D → 3-simplex



cross sectional areas are triangles... 2 simplex

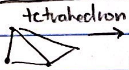
So, we scale each side of triangle by t to get the tetrahedron... $\frac{1}{2}(t)^2$

scaled area = $\frac{t^2}{2}$, so now we add all together using integral

$\int_0^1 \frac{t^2}{2} = \int \frac{t^{3-1}}{(3-1)}$, so our hypervolume is $\frac{1}{k!}$, which is $\frac{1}{6}$ for standard tetrahedron

this checks out!

from 3D to 4D → 4-simplex



→ tricky shape I can't conceptualize (can anyone? is it possible?)

* new shape has tetrahedron as cross-sectional volume... so we are scaling all three sides of our tetrahedron by t...

$\frac{1}{6}$ goes to $\frac{t^3}{6}$

our integral is now $\int_0^1 \frac{t^3}{6}$, which is equivalent to $\int_0^1 \frac{t^{4-1}}{(4-1)!}$,

so again, since this is true, our hypervolume is $\frac{1}{k!} = \frac{1}{24}$

* We see a pattern here... the 2 simplex, 3 simplex, and 4 simplex all have hypervolumes of $1/k!$, so, by induction, the hypervolume of a k-simplex will be $1/k!$

17 Find the work done by the force field

$$(x+y)dx-(2z+1)dy+(y-z)dz$$

as it moves a particle along the curve

$$C=\{(t^2,\ t+1,\ t-1)\mid 0\le t\le 1\}$$

$$F\cdot dr=(x+y)dx-(2z+1)dy+(y-z)dz$$

$$=(t^2+t+1)2t\,dt-(2t-2+1)\,dt+(t+1-t+1)\,dt$$

$$=(2t^3+2t^2+2t+1+2)\,dt$$

$$=(2t^3+2t^2+3)\,dt$$

$$\int_0^1(2t^3+2t^2+3)\,dt$$

$$=\left[2\frac{t^4}{4}+\frac{2t^3}{3}+3t\right]_0^1$$

$$=\frac{2}{4}+\frac{2}{3}+3$$

$$=\frac{25}{6}$$