Dan's Question in Theorem 11.10

Here's the setting: $\alpha_1, \ldots, \alpha_r$ are the *distinct* roots of m, the minimal polynomial for α.
$\rho_1, \ldots, \rho_s : L \to N$ are *distinct* $K(\alpha)$-monomorphisms, and
$\tau_1, \ldots, \tau_r : N \to N$ are distinct K-automorphisms with $\tau_i(\alpha) = \alpha_i$.

Consider $\phi_{ij} = \tau_i \circ \rho_j$. We claimed these functions are all distinct. Dan's question is - Why are they?

The answer turns out to be pretty direct (but Dan says the answer to my question is too, so it's only fair). First consider the case $\tau_i \circ \rho_j$ and $\tau_i \circ \rho_q$ with $j \neq q$. Because $\rho_j \neq \rho_q$, there exists a $\beta \in L$ such that $\rho_j(\beta) \neq \rho_q(\beta)$. Now, because τ_i is injective, $\tau_i \circ \rho_j(\beta) \neq \tau_i \circ \rho_q(\beta)$, and so the functions are different.

Next consider the case $\tau_i \circ \rho_j$ and $\tau_p \circ \rho_q$ with $i \neq p$ (it doesn't matter here if $j = q$ or not). Consider the effects of these functions on α. Because all of the ρs are $K(\alpha)$-monomorphisms, they all take α to itself. So, $\rho_j(\alpha) = \rho_q(\alpha) = \alpha$. This is a good thing because now we find that $\tau_i \circ \rho_j(\alpha) = \alpha_i$ and $\tau_p \circ \rho_q(\alpha) = \alpha_p$ and because $i \neq p$ and the roots are distinct, we thus have that the functions are different.

Those two cases cover the possibilities where the indices are not identical. Therefore in all cases, the functions are different, as claimed. Not too bad.