circles for land surveying. Translated into modern notation, the areas of circles are taken to be $^3/_4 d^2$ and $^1/_{12} c^2$, where *d* equals the diameter and *c* the circumference. The value of π was initially taken to be 3. (Later Liu Hui improved this approximation to 3.14.) The fifth chapter of the book deals with determining the volumes of solid figures; such as prisms, pyramids, the cylinder, and the circular cone for engineers. Its last chapter on right angles *(gou gu)* applies the Pythagorean theorem.

The *Nine Chapters* has at least two general innovations. First, it employs the "Rule of False Position" to solve equations. This rule is a Chinese algebraic invention, although it was also known to ancient Egyptian scribes. An example of this problem-solving technique follows: given $x + {}^1/_3\, x = 8$. Guess that $x = 3$. This gives $3 + 1 = 4$. Since $^4/_8 = {}^1/_2$, the answer or actual x is $2 \times 3 = 6$. The Chinese mainly applied this rule for solving equations of the type $ax = b$. During the Middle Ages Arabic mathematicians transmitted the "Regula Falsae Positionis" to Europe. Second, the *Nine Chapters* recognizes both positive *(zheng)* and negative *(fu)* numbers as solutions to equations. This is the first known appearance of negative quantities in any civilization. (Liu Hiu represented positive and negative numbers by red and black calculating rods, respectively.)

51. From *The Development of Mathematics in China and Japan**

(Problems in the *Nine Chapters*)

YOSHIO MIKAMI

1. From the sixth chapter *jun shu* on pursuit and alligation

"While a good runner walks 100 paces (or *a*), a bad runner goes 60 paces (or *b*). Now the latter goes 100 paces (or *c*) in advance of the former, who then pursues the other. In how many paces will the two come together?"

The answer is: $a \times c \div (a - b) = 250$ paces.

"A hare runs 100 paces (*a*) ahead of a dog. The latter pursues the former for 250 paces (*b*), when the two are 30 paces (*c*) apart. In how many further paces will the dog overtake the hare?"

The answer is: $cb \div (a - c) = 107^1/_7$ paces.

2. From the seventh chapter *yin bu zu* on excess and deficiency

"When buying things in companionship, if each gives 8 pieces, the surplus is 3; if each gives 7, the deficiency is 4. It is required to know the number of persons and the price of the things bought."

**Source:* From Yoshio Mikami, *The Development of Mathematics in China and Japan* (1913), 16 and 18-20. For a fine brief introduction to the *Nine Chapters* see Joseph Needham, *Science and Civilisation in China* (Cambridge: Cambridge University Press, 1975), vol. 3, 24-27.

"When buying hens in companionship, if each gives 9, then 11 will be surplus; and if each gives 6, then 16 will be the deficiency. What will be the number of persons and the price of the hens?"

These problems are equivalent to the solution of the equations

$$y = ax - b \text{ and } y = a'x + b'.$$

The rule for solution is given as follows:

"Arrange the rates (a and a') forwarded by the partners in buying things. What surpasses (b), or is deficient (b'), be each arranged below these rates, and then cross multiply with them. Add the products together, and one gets the *shih*. The surplus and deficiency being added, make the *fa*. If fractional, first make both members equidenominated. Then the *shih* and the *fa* being divided by the difference of the rates, the quotients represent the price of the things bought and the number of persons, respectively."

According to this rule we first form the arrangement (1), from which by cross multiplication follows (2); and then by addition we have (3). Thus

$$(1)\ \begin{matrix} a & a' \\ b & b' \end{matrix} \qquad (2)\ \begin{matrix} ab' & a'b \\ b & b' \end{matrix} \qquad (3)\ \begin{matrix} ab + a'b \\ b + b' \end{matrix}$$

And now we have to take

$$\text{price} = \frac{ab' + ab'}{a - a'}, \quad \text{persons} = \frac{b + b'}{a - a'}.$$

"Of two water-weeds, the one grows 3 feet and the other one foot on the first day. The growth of the first becomes every day half of that of the preceding day, while the other grows twice as much as on the day before. In how many days will the two grow to equal heights?"

This question is solved by applying the method of the surplus and deficiency. For in two days there is a deficiency of 1·5 feet; and in 3 days a surplus of 1·75 feet.

The answer is thus given to be $2\,^6/_{13}$ days, when both grow to the same height of 4 feet and $8\,^6/_{13}$ decimal parts.

3. From the eighth chapter *fang cheng* on the way of calculating by tabulation (This chapter treats simultaneous linear equations using both positive and negative numbers.)

The first problem in this section reads:

"There are three classes of corn, of which three bundles of the first class, two of the second and one of the third make 39 measures. Two of the first, three of the second and one of the third make 34 measures. And one of the first, two of the second and three of the third make 26 measures. How many measures of grain are contained in one bundle of each class?"

"Rule. Arrange the 3, 2 and 1 bundles of the three classes and the 39 measures of their grains at the right. Arrange other conditions at the middle and at the left."

The arrangement then takes the form shown in the accompanying diagram.

1	2	3	1st class
2	3	2	2nd "
3	1	1	3rd "
26	34	39	measures

The text proceeds: "With the first class in the right column multiply currently the middle column, and directly leave out."

This means to subtract the terms in the right column as often as possible from the corresponding terms of the middle column thus multiplied. The arrangement after such an operation becomes as annexed.

1	0	3	1st class
2	5	2	2nd "
3	1	1	3rd "
26	24	39	measures

"Again multiply the next, and directly leave out."

This teaches to repeat the operation with the left column. The arrangement now becomes as represented in the diagram.

0	0	3	1st class
4	5	2	2nd "
8	1	1	3rd "
39	24	39	measures

"Then with what remains of the second class in the middle column, directly leave out."

That is, the 2nd class from the left column is to be eliminated by applying the process as above described. The result is as shown here.

		36	third class
		99	measures

"Of the quantities that do not vanish, make the upper the *fa*, the divisor, and the lower the *shih*, the dividend, i. e., the dividend for the third class.

"To find the second class, with the divisor multiply the measure in the middle column and leave out of it the dividend for the third class. The remainder, being divided by the number of bundles of the second class, gives the dividend for the 2nd class.

"To find the first class, also with the divisor multiply the measures in the right column and leave out from it the dividends for the third and second classes. The remainder being divided by the number of bundles of the first class, gives the dividend for the first class.

"Divide the dividends of the three classes by the divisor, and we get their respective measures."

The above process, as will be seen on a first glance, does not deviate seriously from our procedure in solving the simultaneous system

$$3x + 2y + 1z = 39,$$
$$2x + 3y + 1z = 34,$$
$$1x + 2y + 3z = 26.$$

The only difference is that the expressions are arranged in vertical columns instead of writing in horizontal lines as we do nowadays. The Chinese manipulation appears however without doubt to have been carried on with the calculating pieces, not in a written scheme.

The answer for the above problem is given as $9\,{}^{1}/_{4}$, $4\,{}^{1}/_{4}$, and $2\,{}^{3}/_{4}$ measures of grain, respectively.

The above process is directly applicable only in the case when the terms in a column are all subtractible from other columns. But how had done the ancient Chinese to proceed in the case, for which such is not the case?

The following is one of such examples.

"There are three kinds of corn. The grains contained in two, three and four bundles, respectively, of these three classes of corn, are not sufficient to make a whole measure. If however we add to them one bundle of the 2nd, 3rd, and 1st classes, respectively, then the grains would become full one measure in each case. How many measures of grain does then each one bundle of the different classes contain?"

1		2	1st class
	3	1	2nd "
4	1		3rd "
1	1	1	measures

The arrangement that corresponds to this problem will be as annexed.

The empty blanks have no numbers to be arranged therein.

The answer is given as ${}^{9}/_{25}$, ${}^{7}/_{25}$, ${}^{4}/_{25}$ measures of the three classes.