1

The Ancient World

1.1 Prehistory

When anthropologists speak of a group's culture, they mean the sum of all the human activity of the group: how they talk, think, eat, play, and do mathematics. Every culture in the world creates some sort of mathematics, though it may be on a very basic level. The most ancient evidence of mathematical activity comes from a wolf's bone on which fifty-five notches are carved, grouped in sets of five. The bone was found in the Czech Republic and is about 35,000 years old. A similar artifact was found at Ishango, on the shores of Lake Edward in Zaire. The Ishango bone (from a baboon) dates to around 18,000 B.C., and is particularly intriguing since along one side, the notches are grouped into sets of 11, 13, 17, and 19, suggesting an interest in prime numbers.

At that time, mankind obtained food by hunting animals and gathering plants. But about ten thousand years ago, an unknown group of people invented agriculture. This forced people to establish permanent settlements so the plants could be cared for until harvest: the first cities.

The culture of city-dwellers is called *civilization* (from *civilis*, "city-dweller" in Latin). Since cities were originally farming communities, they were established in regions that provided the two fundamental needs of agriculture: fertile soil and a reliable water supply. River flood-plains provide both, and the most ancient civilizations developed around them.

Figure 1.1. The Ishango Bone. Photograph courtesy of Science Museum of Brussels.

1.2 Egypt

In Egypt, civilization developed around the Nile river, whose annual flooding deposited silt that fertilized the fields of the Egyptian farmers. As early as 4000 B.C. the Egyptians may have noticed that it took approximately 365 days from one Nile flood to the next. The Egyptian calendar divided the year into 12 months of 30 days, with 5 extra days celebrated as the birthdays of the main gods of the Egyptian pantheon. These days of religious festivities were the original *holidays* ("Holy days").

Egypt is divided into two main parts, Lower Egypt and Upper Egypt. Lower Egypt consists of the marshy areas where the Nile empties into the Mediterranean, known as the Delta (because of its resemblance to the Greek letter Δ, pointing southwards); the term would later be applied to *any* river's outlet into a sea or lake. Upper Egypt is formed by the region between the Delta and the First Cataract, or set of waterfalls, at Aswan. According to tradition, Upper and Lower Egypt were united around 3100 B.C. by Narmer, whom the Greeks called Menes. Archaeological evidence points to an even earlier King Scorpion who ruled over a united kingdom, and recently the tombs of King Scorpion and his successors have been identified.

The kings of Egypt were known as *pharaohs* ("Great House," referring to the Royal Palace). The pharaohs were revered as gods, though this did not spare them from being criticized, plotted against, and deposed. Though the rulers changed, the fundamental in-

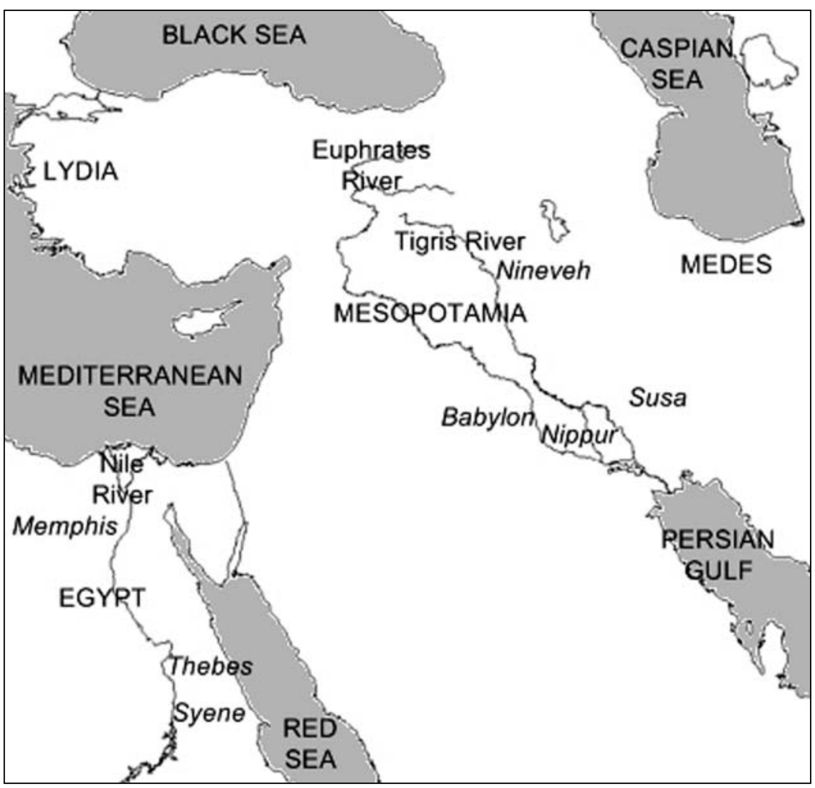


Figure 1.2. Egypt and the Fertile Crescent

stitutions of pharaonic Egypt would last for nearly three thousand years, a period of time usually referred to as Dynastic Egypt.

Dynastic Egypt is traditionally divided into thirty dynasties, according to a division first described by the Greek-Egyptian scholar Manetho around 280 B.C. These thirty dynasties are grouped into five periods: the Old Kingdom (roughly 3100 B.C. to 2200 B.C.); a First Interregnum (2200 B.C. to 2100 B.C.); a Middle Kingdom (2100 B.C. to 1788 B.C.); a Second Interregnum (1788 B.C. to 1580 B.C.); and a New Kingdom (1580 B.C. to 1090 B.C.).[1] Manetho's division of the reigns has been retained, though his dates are no longer considered accurate; indeed, the dates of ancient Egypt are very uncertain and Egyptologists themselves differ by up to a hundred years on the dates of the reigns of early pharaohs.

The best-known features of Egyptian civilization are the pyramids. The step pyramid at Sakkara, near the Old Kingdom capital of Memphis, is the most ancient; it was completed under the direction of the architect Imhotep around 2650 B.C. The largest and best known pyramid is the Great Pyramid at Gizeh, finished around 2500 B.C. and the oldest of the Seven Wonders of the World. Moreover, it is the *only* one of the Seven Wonders still standing, a tribute to the skill of the Egyptian builders and the dry climate of Egypt.

By 2700 B.C., a form of writing had been invented. Since many examples were found adorning the walls of Egyptian temples, it was erroneously believed that the writings were religious in nature. Hence this form of Egyptian writing became known as *hieroglyphic* ("sacred writing" in Greek).

Hieroglyphs may have originally been pictograms (small pictures of the object being represented, such as a set of wavy lines $\approx$ to represent a river) or ideograms (stylized figures that represent an abstract concept, such as the emoticon `;-)` to indicate "just kidding"). However, this form of writing would require the knowledge of hundreds, if not thousands, of symbols to represent the common words in a language, so hieroglyphs soon took on new meanings as *sounds*. For example, we might draw $\approx$ to represent the word "river" or to refer to a man named Rivers, but in time $\approx$ might come to represent the sound "riv-" or even the initial sound "r-."

1.2.1 Egyptian Mathematics

To write numbers, we could spell them out: two hundred forty-one. However, writing out number words is tedious and all civilizations have developed special symbols for numbers. The use of a vertical or horizontal stroke to represent "one" is nearly universal, as is the use of multiple strokes to represent larger numbers: thus, "three" would be |||. Of course, such notation rapidly gets out of hand: try to distinguish ||||||||||||| from ||||||||||||. The next logical step would be to make a new symbol for a larger unit. In hieroglyphic, ∩ represents ten, 𓍢 represents one hundred, 𓆼 represents one thousand, and other symbols were used for ten thousand, one hundred thousand, and one million. To indicate a large number, the symbols would be repeated as many times as necessary: two hundred forty-one would be |∩∩∩∩𓍢𓍢 (note that the Egyptians wrote from right to left, and placed the greatest values first). Since the value of the number is found by adding the values of the symbols, this type of numeration is called additive notation.

[1] Menes is the first pharaoh of the First Dynasty. King Scorpion and his immediate successors, who predate Menes, are collectively grouped into Dynasty Zero.

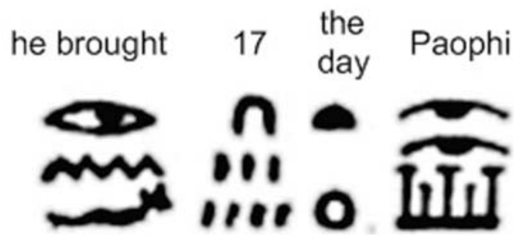
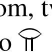
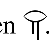


Figure 1.3. From the Rosetta Stone, line 10. Inscription reads, "On the 17th day of Paophi, he [Ptolemy V] brought...."

Fractions were generally expressed as sums of unit fractions. Thus $\frac{2}{5}$ would be written as $\frac{1}{15}$ and $\frac{1}{3}$. These unit fractions were written by placing a symbol,  , read as *ro* (an open mouth), above the hieroglyphic representation of the denominator. Thus $\frac{2}{5}$ would be  .

Some common (non-unit) fractions had special symbols, though the only one used with any consistency was a symbol for two-thirds. The immense length of Egyptian history can be illustrated by the two thousand year history of this symbol: in the Old Kingdom, two-thirds was written as  but a thousand years later, this form would change into  . It would be another thousand years before this form was written .

By 2600 B.C., a cursive form of hieroglyphic, suitable for writing on softer materials, came into existence: hieratic. All Egyptian mathematical treatises are written in hieratic, on papyrus, cloth, or leather. Unfortunately, these materials disintegrate rapidly when exposed to water, insects, and sunlight, so our knowledge of Old Kingdom mathematics is limited to indirect evidence (like the existence of the pyramids or an accurate calendar) and hieroglyphic numerals.

The oldest mathematical texts date to the Middle Kingdom, six hundred years after the construction of the Great Pyramid of Gizeh. The Reisner papyrus, named after its discoverer, dates to the reign of the Twelfth Dynasty pharaoh Sesostris (around 1900 B.C.) who established a chain of forts south of Aswan to keep Egypt's borders secure against the Nubians. Sesostris also sponsored a number of building projects, including additions to the great temple complex at Karnak (near Thebes, the capital of the Middle Kingdom).

The Reisner papyrus is a set of four worm-eaten rolls, and appears to be a set of books for a construction site. The readable portions include a list of employees, as well as calculations of volumes and areas. A typical computation is determining the number of workmen needed to excavate a tomb, given the dimensions of the pit and the expected volume of dirt to be removed by a workman each day (apparently 10 cubic cubits, or roughly 30 cubic feet).

Sesostris's dynasty ended with the reign of Sebeknefru (around 1760 B.C.), the earliest female ruler whose existence is definitively established. Her reign was peaceful, but her death touched off a civil war and over the next century, nearly seventy pharaohs would rule in quick succession.

The Moscow or Golenishchev papyrus (named after its current location or discoverer) was written during this era. It includes a number of geometric problems. For example, Problem 7 asks to find the dimensions of a triangle with area 20 *setat*, where the ratio of the height to the base is 2 1/2 to 1. The scribe's solution is to double the area (making a rectangle with sides in ratio 2 1/2 to 1) then multiply by 2 1/2: this produces a square (of area 100), whose side can be found by taking the square root; this is also the triangle's height. Dividing by 2 1/2 gives the base. Problem 9 finds the surface area of a basket (possibly a hemisphere or a half-cylinder).

Problem 14 contains what is perhaps the greatest mathematical discovery of the ancient Egyptians: the computation of the exact volume of the frustum of a pyramid. The idea of a mathematical formula was as yet non-existent; the student was expected to generalize from the given example. Thus the Moscow papyrus gives:

Problem 1.1. *Find the volume for a frustum* 6 *cubits high with [square] base* 4 *cubits and [square] top* 2 *cubits. Square the base to get* 16*; multiply [top side]* 2 *by [bottom side]* 4 *to get* 8*; square the top* 2 *to get* 4*; add these together to get* 28*. Multiply the height* 6 *by* 1/3*, to get* 2*; multiply* 2 *by* 28 *to get* 56*, the volume.*

Thus if the pyramid has a square base of side a, square top of side b, and height h then its volume will be given by $V = \frac{1}{3}h\left(a^2 + ab + b^2\right)$.

Shortly after the Moscow papyrus was written, Egypt was invaded by a mixed group of tribes, consisting primarily of Semites from Palestine and Hurrians from Asia Minor (modern-day Turkey). These invaders swept away all opposition by using a new weapon of war: the horse and chariot. They conquered Egypt but kept its political and religious institutions intact, establishing themselves around 1680 B.C. as the Fifteenth Dynasty. The Egyptians named them the *Hyksos* ("Rulers of Foreign Lands").

Our first complete text of Egyptian mathematics dates to this period, a thousand years after the building of the first pyramids. The Rhind papyrus (named after its discoverer) was, according to the preface, written down in the fourth month of the Inundation Season in the thirty-third year of the reign of A-user-Re by the scribe A'H-MOSÈ (fl. 1650 B.C.?). The introduction to the Rhind papyrus highlights an important problem of history: precise dating of an event. Often we have a better idea of the day and month than for the year itself: because of the reference to the fourth month of the Inundation Season, we know that the Rhind was copied around September. But we do not know A-user-Re's dates with any certainty, so the best guess for the thirty-third year of the reign of A-user-Re is around 1650 B.C.

A'h-mosè claimed the Rhind was based on an older work, dating back to the Thirteenth Dynasty pharaoh Ne-ma'et-Re and thus contemporaneous with the Moscow papyrus, a claim that might or might not be true. Why would someone deny that their work was original? In the days before printing, all texts had to be copied by hand: they were literally *manuscripts* ("hand written" in Latin). Since it takes just as much effort to copy a worthless text as a worthwhile one, this meant that the only texts that were copied and recopied were the most important. One way to give your work a veneer of importance is to attribute it to some great author of the past or some time period that people looked back to with nostalgia: hence, the first five books of the Old Testament are attributed to Moses, even though they were not written down until a thousand years after the events they claim to describe.

The first part of the Rhind papyrus consists of the quotient of 2 and the odd numbers from 3 to 101. From this, we can discern many details of how the Egyptians performed calculations. A typical computation is 2 divided by 13, whose quotient is $\frac{1}{8} + \frac{1}{52} + \frac{1}{104}$.

The computation proceeds as follows:

	1	13		
	1/2	6	1/2	
	1/4	3	1/4	
\	1/8	1	1/2	1/8
\	4	52	1/4	
\	8	104	1/8	

The second, third, and fourth lines seem to represent the product of 13 and $1/2$, $1/4$, and $1/8$, respectively. The fifth line seems to refer to the fact that $13 \cdot 4 = 52$ implies $13 \cdot \frac{1}{52} = \frac{1}{4}$. Likewise, the last line obtains $13 \cdot \frac{1}{104} = \frac{1}{8}$ from $13 \cdot 8 = 104$. The lines indicated with the \ can then be interpreted to read:

$$13 \cdot \left(\frac{1}{8} + \frac{1}{52} + \frac{1}{104} \right) = 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{4} + \frac{1}{8}$$

Hence $13 \cdot \left(\frac{1}{8} + \frac{1}{52} + \frac{1}{104} \right) = 2$, and thus 2 divided by 13 is $\frac{1}{8} + \frac{1}{52} + \frac{1}{104}$.

In the Rhind papyrus, A'h-mosè shows how to solve simple linear problems using a variety of methods, including the method of false position. The student was expected to generalize from the examples, which began with:

Problem 1.2. *A number and its seventh make* 19 *[i.e.,* $x + \frac{1}{7}x = 19$*]. Suppose* 7*; its seventh is* 1*, and together* 8*. Divide* 19 *by* 8*, then multiply by* 7*. Solution:* 16 *and* $\frac{1}{2}$ *and* $\frac{1}{8}$.

To find the area of a circle, A'h-mosè gave the example:

Problem 1.3. *Find the area of a circular plot of land with a diameter of* 9 khet. *Take away one-ninth of the diameter, leaving* 8*; multiply* 8 *by itself to get the area,* 64 setat.

The volume of a cylinder was computed by multiplying the area of the base (using the above method) by the height.

The problems in the Rhind are all linear, though the contemporary Kahun papyrus seems to suggest the Egyptians could also solve non-linear equations. Unfortunately the problem statement is missing, and must be reconstructed from the method of solution.

1.2.2 The New Kingdom

The Hyksos conquest of Egypt depended on advanced technology: the horse, chariot, and compound bow. But all of these were soon duplicated by native Egyptians, who made ready to throw out their masters. A revolt began in Thebes, and by 1521 B.C., the Hyksos were expelled and a new dynasty, the eighteenth, was formed. Mathematicians might smile at the name of the first pharaoh of the new dynasty: Ahmose.[2] Ahmose went on to conquer Nubia, to the south, and his successors established a vast Egyptian Empire that ultimately extended well into Palestine. The Kingdom of Egypt underwent a great cultural renaissance.

One of the critical events of the Eighteenth Dynasty concerned religion. For millennia, the Egyptians were polytheists, revering many gods, though chief among them was Amon. Around 1350 B.C., the pharaoh Amenhotep IV ("Amon is satisfied") became a monotheist,

[2]The name was a popular one during and immediately after the Hyksos era.

worshipping a single god. In Amenhotep's case, the single god was the sun god Aton, and he renamed himself Akhenaton ("One Useful to Aton"). Akhenaton established a new capital at Tell-el-Amarna, where Egyptian art and sculpture flourished.

Akhenaton attempted to eliminate the worship of other gods besides Aton, and in this he failed. His appointed successor Smenkhkare had a very short reign, and was succeeded by an eight-year-old boy named Tutankhaton. With the accession of a minor, the polytheists saw the chance to return Egypt to the old religion. They forced the young pharaoh to restore Amon to the position of chief among many gods, and to rename himself Tutankhamun ("Living Image of Amon"). Tutankhamun's unplundered tomb was discovered in 1922 by Howard Carter and the fifth Earl of Carnarvon, George Edward Stanhope. A few months later Stanhope died suddenly, giving rise to a belief in a "curse of the pharaohs," though Carter, the first to actually enter the tomb, survived another seventeen years.

The internal religious warfare weakened the Eighteenth Dynasty and led to the rise of the most famous Egyptian dynasty of all, thanks to its pharaoh Rameses the Great. Rameses, the third pharaoh of the Nineteenth Dynasty, built enormous monuments to himself all over Egypt. The most spectacular was at Abu Simbel, where four 67-foot-high statues of a seated Rameses loom beside the entrance to a temple. In the 1960s, the damming of the Nile at Aswan formed a lake that threatened to submerge the temples, so they were cut apart and reassembled on higher ground.

The Berlin papyrus was written during the Nineteenth Dynasty and gives the first clear example of a non-linear problem solved by the ancient Egyptians. The problem is:

Problem 1.4. *A square with an area of* 100 *square cubits is to be divided into two squares whose sides are in ratio* 1 *to* $\frac{1}{2}\ \frac{1}{4}$ *[i.e.,* 1 *to* $\frac{1}{2}+\frac{1}{4}$*].*

The scribe's solution is by means of false position: supposing the side of the bigger square is 1, making the side of the smaller square $\frac{3}{4}$, the total area will be 1 $\frac{9}{16}$, which is the area of a square of side 1 $\frac{1}{4}$. Divide 10 by 1 $\frac{1}{4}$ to get 8, then multiply the initial guess (1) by 8 to yield the actual side of the larger square; the side of the smaller square is thus $\frac{3}{4}$ of 8 or 6.

After Rameses, the Egyptian Empire underwent a long and final decline. The reasons are varied, but one factor may have contributed more than anything else. The Egyptian Empire was conquered and controlled by warriors using bronze. But around the reign of Rameses the Great and the writing of the Berlin papyrus, the Hittites, an obscure tribe living in the foothills of the Caucasus mountains, learned how to refine iron from its ores. An iron equipped army could easily destroy one outfitted with bronze. The Egyptians adopted the horses, chariots, and bows of the Hyksos, and might have adopted the iron technology of the Hittites as well—but there are few sources of iron ore in Egypt.

1.3 Mesopotamia

Egypt is the westernmost of the ancient river civilizations, and is just south of one end of the Fertile Crescent, a region that includes modern Iraq, southeastern Turkey, Syria, Israel, Lebanon and Jordan. A second great river civilization arose at the eastern end of the Fertile Crescent, in modern Iraq. Located between the Tigris and Euphrates rivers, it became known as Mesopotamia ("between the rivers" in Greek). Today, the Tigris and

Euphrates merge about a hundred miles inland of the Persian Gulf, though in ancient times, the two rivers flowed separately into the sea.

Most historians believe that agriculture was invented in Mesopotamia, and thus civilization began in this land between the rivers. The most ancient civilization in this region was the Sumerian, and by 3000 B.C., a group of Sumerian cities existed in Lower Mesopotamia, the region closer to the Persian Gulf. Unlike the contemporary Egyptians, the Sumerians never united into a single kingdom, and each city was a separate political entity with its own customs, army, and foreign policy; hence they are generally referred to as city-states.

Like the Nile, the Tigris and Euphrates flood periodically. But while the flooding of the Nile is a gentle, welcome event that signals the beginning of the year and a renewal of life, Mesopotamian floods are often catastrophic events. One flood tale was included in the story of Gilgamesh, the King of Uruk (a city along the Euphrates). When a close friend dies, Gilgamesh searches the world for the secret of immortality, and comes across the one man who has found it: Utnapishtim. He tells Gilgamesh the story of how the world became so populated that the gods, tired of hearing the racket made by so many people, decided to exterminate mankind with a great flood. One of the gods, Ea, warned Utnapishtim of what was to happen, and had him build a boat in which to save himself and his family. To find when the waters had receded enough to land, Utnapishtim released a dove, which could find no resting place and thus returned to the boat, and later a raven, which found a roost and never returned. The legend of Gilgamesh is almost certainly one of the inspirations of the Biblical story of Noah. As for immortality, Utnapishtim explained to Gilgamesh how to find the plant that confers immortality, but the plant is eaten by a snake while Gilgamesh sleeps. Again, the parallels to the Biblical story of how mankind was deprived of immortality through the actions of a serpent suggest that the ancient Hebrews drew heavily upon these legends.

1.3.1 Positional Notation

The Egyptians wrote on papyrus, leather, cloth, or stone. The Mesopotamians had none of these in abundance; instead, they had mud. Fortunately, mud could be written on with a stylus while still wet, and then baked into a brick-hard clay tablet. Such tablets can last for millennia, virtually unchanged. In contrast to a handful of Egyptian papyri, we have thousands of clay tablets.

Mesopotamian script consists of *cuneiform* ("wedge-shaped" in Greek) symbols, distinguished by the number of marks and their orientation. Deciphering an arbitrary text with no knowledge of the content would be virtually impossible, but if the text is mathematical, we can make considerable headway. Consider the table text reproduced in Figure 1.4. As you read down the rows of the first and second vertical columns, the number of 𒁹 symbols increases from one to nine, after which a new symbol, 𒌋, appears. It seems reasonable to suppose that 𒁹 indicates a unit, 𒈫 indicates two units, and so on, up until 𒌋 indicates ten units; then 𒌋 𒁹 indicates eleven and so on. Mesopotamian numeration *appears* to be additive.

However, consider the third column. The first entry is obliterated, but if our assumption that 𒁹 represents one and 𒌋 represents ten is correct, then the second through seventh rows read: four, nine, sixteen, twenty-five, thirty-six, and forty-nine. These are the squares of

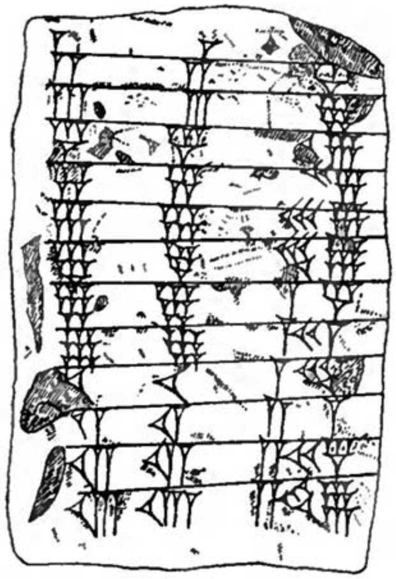


Figure 1.4. A Table Text from Nippur.

the numbers in the second column, so the eighth entry ought to represent sixty-four. So how shall we interpret the fact that the number is written using five 𒁹s? If there is any consistency in the table text, then one of the 𒁹s must represent sixty and the remaining four must represent four. This suggests that Mesopotamian notation was positional, base 60.

There are some drawbacks in the positional notation used in Mesopotamia. First, there was no way to indicate the order of magnitude: 𒁹 by itself might mean one unit, or it might mean one sixty, or it might even mean one sixtieth. Context was the only way to decide between the possible interpretations.

Another problem was that there was no way to indicate the *lack* of an order of magnitude. For example, consider the eleventh line of the table text. The number in the third column should be the square of eleven, or one hundred twenty-one. This is two sixties and one unit, and the scribe has properly recorded the number, leaving a wide space between the 𒁹s representing the sixties and the 𒁹 representing the unit. But a careless or inexperienced scribe might not leave enough space between the orders of magnitude, making it possible to read this as three 𒁹s: three units or possibly three sixties.

Our own system uses a number of devices to solve these problems. First, we have nine different quantity symbols (1 through 9), so that adjacent symbols actually represent different orders of magnitude: 123 represents 3 units, 2 tens, and 1 hundred. Moreover, we use a decimal point to separate units from fractions of a unit, while a tenth symbol (0) indicates the lack of an order of magnitude. Thus 10.1 is 1 ten, no units, and 1 tenth. In modern transcriptions of Babylonian sexagesimals, the comma is used to separate the orders of magnitude and the semicolon (;) separates the units from the fractions; the zero is used freely. Thus 1 sixty and 1 sixtieth would be written $1, 0; 1$. However (and this is important to emphasize), the Mesopotamians *never* indicated the sexagesimal point, and it was not until very late in their history that a symbol for an empty space was used.

In 1898 an American expedition began excavating the city of Nippur, the center of worship of the Sumerian god Enlil ("Lord Wind"), and thus one of the main religious centers in Mesopotamia. There they found thousands of mathematical cuneiform texts in the eastern section of the city (known as the scribal quarter). Originally attributed to being from a "temple library", though later expeditions cast doubt on this claim, these tablets are of the type known as table texts, and may have been written as early as 2200 B.C. These are essentially multiplication tables, and provide many examples of Mesopotamian numeration; there are enough careless errors on the tablets to suggest they were written by apprentice scribes.

One might expect the multiplication tables to include the multiples of 1, 2, 3, and so on, and most of the tablets do. But tables exist for the products of (in sexagesimal) $3, 20$ or even $44, 26, 40$. Such tables might be explained as "make work" assigned by teachers to keep students busy, but a more likely explanation is found by noting that dividing by a number like 18 is the same as multiplying by $\frac{1}{18}$, and in sexagesimal, $\frac{1}{18} = 0; 3, 20$. For $44, 26, 40$ we note that $0; 0, 44, 26, 40 = \frac{1}{81}$. Thus these tablets were probably used for division problems.

1.3.2 Babylon

The disunity of the Sumerican city-states made them easy prey for any unified conqueror. Under Sargon of Agade (or Akkad), a group of tribes speaking a Semitic language conquered the Sumerians of Mesopotamia around 2300 B.C. Since Sargon and the Akkadians ruled over a diverse population of which they themselves were only a small minority, we speak of the Akkadian *Empire*. It was the first empire in history. It lasted less than a century, and by 2200 B.C., the empire was destroyed and the capital city of Akkad was so devastated that even today its exact location is unknown. But the brief existence of the Akkadian Empire had some lasting consequences. One of the main results was that Sumerian (a language unrelated to any others) gradually disappeared as a spoken language, to be replaced with Semitic languages (such as Akkadian). This linguistic shift affected place names, and an obscure Sumerian city, *Ka-dingr* ("The Gate of God" in Sumerian) was translated literally into Akkadian as *Bab-ilu*, a name that eventually became Babylon.

Around 1800 B.C. the Babylonians conquered all of Mesopotamia. Since most of the problem texts seem to date to around this time, Mesopotamian mathematics is frequently referred to as Babylonian mathematics.

The most famous Babylonian king was Hammurabi, who reigned around 1750 B.C. Hammurabi achieved lasting fame for having inscribed on a *stele* (stone pillar) a complex and sophisticated legal code. This was not the first written code of laws, though it is one of the earliest that we have in its entirety.

The stele was originally in Sippar, at the temple of the sun god Shamash. At the top is a depiction of Hammurabi receiving the laws from Shamash. Flames emerge from the god's shoulders, which is suggestive of the much later Biblical myth of Moses receiving the ten commandments from a flaming god. The code of Hammurabi divides people into several classes, including noblemen, commoners, and slaves. The law is primarily punitive and consists of prescriptions: if X occurs, the penalty is Y. Many crimes are punishable by death or mutilation, but monetary fines are also common. The legal theory of Hammurabi's empire gave the classes distinct responsibilities and punishments: in general, harm to a noble warranted the greatest punishment, and harm to a slave the least. But nobles were also held to a higher standard of behavior, and misconduct by a noble was more severely punished: if a noble stole livestock (as opposed to other types of property), they had to pay 30 times the value, but a commoner only 10 times. The code also regulates prices and wages. The law code must have made a great impression on the people of the time, for shortly after Hammurabi's death, an invading army carried the stele away to Susa, where it was discovered in 1901 by French archaeologists.

We cannot compare the legal codes of Babylon and Egypt, for the surviving Egyptian law codes date to a time much later than Hammurabi. We may, however, compare their mathematics, for the Rhind papyrus and the Babylonian problem texts are contemporaneous. Many Babylonian problems dealt with canals, a necessity of life in Mesopotamia, required both for irrigation and flood control. A typical problem was:

Problem 1.5. *A canal* 5 *GAR long,* $1\frac{1}{2}$ *GAR wide, and* $\frac{1}{2}$ *GAR deep is to be dug. Each worker is assigned to dig* 10 *GIN, and is paid* 6 *SE. Find the area, volume, number of workers, and total cost.*

where GAR, GIN, and SE are units of quantity (the GIN is equal to a cubic GAR, for example).

The Babylonians frequently ventured beyond simple linear equations. Many canal problems resulted in quadratic equations, though a few "pure math" problems were posed:

Problem 1.6. *The* igibum *exceeded the* igum *by* 7*, and the product of the two is* 1, 0*. Find the* igibum *and the* igum*.*

The problem is equivalent to solving $x(x+7) = 60$, and the scribe's solution relied on an identity we would write as

$$\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = ab$$

Thus given the product ab and $a \pm b$, we can find $a \mp b$. Finally, the *igibum* and the *igum* can be found using

$$\frac{a+b}{2} + \frac{a-b}{2} = a$$
$$\frac{a+b}{2} - \frac{a-b}{2} = b$$

There is some evidence that degenerate cubics with rational solutions were solved by transformation into a standard form and referring to tables of values. Even approximate solutions to exponential equations were found using linear interpolation (also known as the method of double false position or the method of secants).

Much of our knowledge of Babylonian geometry comes from a French expedition to Susa, about 200 miles east of Babylon. In 1936, the expedition uncovered a horde of cuneiform texts, including some on geometrical subjects, which seem to be contemporaneous with the arrival of Hammurabi's stele. One tablet shows a circle with the cuneiform numbers 3 and 9 on the circumference and 45 on the inside. This suggests that the area of a circle was found by taking the circumference, 3, squaring it to get 9, then dividing by 12 to get 0; 45. This gives an even worse approximation for the area of a circle than the one used by the Egyptian scribes. This might suggest the Mesopotamians were poor geometers, though other tablets of the Babylonian era show they knew of and used the Pythagorean Theorem; the most remarkable is a tablet that seems to indicate the diagonal of a square is 1; 24, 51, 10 of its side which, if converted to its decimal equivalent, gives $\sqrt{2} \approx 1.414213\ldots$.

In addition, Plimpton 322, believed to have been written around Hammurabi's time, gives a list of 15 rational numbers that satisfy $a^2 + b^2 = c^2$, which suggests an early interest in number theory, and might point to the knowledge of the Pythagorean Theorem as well. Another tablet gives relationships between the area A_n of a regular n-gon with a side length of S_n as:

$$A_5 = 1;40S_5^2 \qquad A_6 = 2;37,30S_6^2 \qquad A_7 = 3;41S_7^2$$

These values are accurate to within about 3%. Moreover, if C is the circumference of the circle circumscribed about a hexagon, then the tablet also suggests $S_6 = 0;57,36C$, accurate to within 1%.

1.3.3 The Iron Empires

Around 1400 B.C., the Hittites of Asia Minor learned how to smelt iron. Before they could turn this to their advantage, they were conquered by the Assyrians, who were the first to field an army of iron.

In the ancient world, a conquered population could expect death or enslavement. The Assyrians added a new horror to war, and routinely tortured and mutilated their captives in what can only be called a calculated attempt to terrify potential adversaries into submission. The end result of this policy was that the Assyrian empire, established by force, had to be maintained by force and was ultimately destroyed by force.

In 612 B.C., the Assyrian capital of Nineveh was utterly destroyed by an alliance that included the Medes under Cyaxeres and the Chaldeans under Nabu-apal-usur ("Nabu guards the prince"), usually known as Nabopolassar. Under him the Chaldeans rose to become a mighty empire. His son, Nabu-kudurri-usur ("Nabu protects my boundaries") rebuilt Babylon and turned it into the capital of a new empire, properly called the Chaldean Empire but also referred to as the Babylonian or Neo-Babylonian Empire.

Nabu-kudurri-usur is better known as the Nebuchadnezzar of the Bible (though we will use the more correct spelling Nebuchd*r*ezzar). According to tradition, Nebuchadrezzar married a Medean princess, but she became so homesick for her homeland in the foothills of Asia Minor that he had built for her a fabulous tiered garden. The "Hanging Gardens of Babylon" became one of the Seven Wonders of the World.

In 598 B.C., Judea revolted against Babylonian rule. Nebuchadrezzar suppressed the revolt, deported some of the leading citizens to Babylonia, but left the kingdom, temple, and local government intact. A second revolt began. Nebuchadrezzar captured Jerusalem in 586 B.C., destroyed the Temple of Solomon, and arranged for a second deportation. This began the era of Jewish history known as the Babylonian Captivity.

Meanwhile, the Medes were moving into Asia Minor, where they met the expanding Lydian Empire. Five years of inconclusive warfare between the Lydians and the Medes were brought to an abrupt end when, during a battle, “day turned into night:” a solar eclipse occurred. The opposing commanders took this as a sign of the displeasure of the gods, and signed a hasty peace. We can calculate that a total eclipse was visible from Asia Minor on May 28, 585 B.C., so this battle is the earliest historical event that can be given an exact date.

According to one story, the eclipse was predicted by the first mathematician we know by name: Thales of Miletus.

For Further Reading

For the history of Egypt and the Fertile Crescent, see [1, 18, 60, 68, 106, 130]. For the mathematics and science of the region, see [22, 41, 52, 84, 85].

2

The Classical World

2.1 The Greeks

Greece is a mountainous land of limited fertility, so settlements tended to cluster in the narrow valleys and on the various peninsulas that jut out into the Mediterranean. The relative isolation of the settlements encouraged them to form independent city-states. Like Sumeria, the individual Greek city-states were no match for a united conqueror, so by the time of A'h-mosè, mainland Greece was part of the Minoan Empire, centered on the island of Crete. The Minoan royal palace at Knossos was an impressive structure. Spread over six acres of land, it contained hundreds of rooms—and flush plumbing. Around 1500 B.C., the volcanic island of Thera exploded, causing a tsunami so devastating that the Minoans never recovered; this was probably the origin of the legend of Atlantis, retold by Plato a thousand years later.

The tsunami weakened the Minoans and the Greeks revolted. They invaded Crete, burned the palace and other centers of civilization, and lost the secret of flush plumbing for thousands of years. The Greeks established the Mycenaean Empire, named after one of their main cities. Around 1250 B.C., the Mycenaens besieged and destroyed Troy, a city on the coast of Asia Minor. But shortly after, the bronze-equipped conquerors of Troy were overcome by iron-wielding Dorians from the north. The Dorian dialect of Greek was difficult for the Mycenaeans to understand; they lampooned Dorian speech as "bar-bar" (much as we might describe someone's ramblings as "yadda-yadda") and called them *barbarians*, a term that has since been applied to cultures that have no permanent cities. Since it is easy (and usually incorrect) to associate "uncivilized" with "simple-minded," a particularly plain type of architectural column was later called *Doric* in the (mistaken) belief that they were products of Dorian builders.

By 700 B.C., Greece had recovered from the Dorian invasions, and the population was booming. To relieve crowding, a city-state would establish colonies around the Mediterranean. The "mother city," or *metropolis* in Greek, would equip a colony with people, ships, and government. As often as not, the colonies became independent of the mother city within a few generations, though some kept close ties.

These colonies went out in two directions, which can lead to confusion among students of classical geography. The Greek settlers who went west, towards the Italian peninsula and the island of Sicily, were the first Greeks encountered by the Romans. Thus the Romans referred to southern Italy as *Magna Graecia*, Latin for "Greater Greece" (where in this context, "Greater" means "Larger").

Other Greek colonists went east, to Asia Minor. Many who settled there came from the shores of the Ionian Sea, on the west coast of modern Greece. In commemoration of their original homeland, the Greek colonies in Asia Minor were collectively known as Ionia, even though they nowhere border the Ionian Sea.

2.1.1 Thales and Pythagoras

THALES (fl. 6th cent. B.C.) lived in Miletus, in Ionia. He is credited with having "discovered" five geometrical propositions (which suggests that he did not prove them):

1. A circle is bisected by its diameter.
2. The base angles of an isosceles triangle are equal.
3. Vertical angles are equal.
4. If, in a triangle, a side and its two adjacent angles are equal to the side and two adjacent angles of another triangle, then the two triangles are equal.[1]
5. An angle inscribed in a semicircle is a right angle.

There is a story that Thales sacrificed an ox upon his discovery of the last theorem.

The life of Thales is well documented in the *Histories* of Herodotus, who lived in the fifth century B.C. Herodotus has been called the "father of history," because he was the first to apply recognizably modern historical methods to the problem of the past, as well as the "father of falsehoods," because so much of what he concludes was fantastical and hard to believe. However, the more fantastical statements were usually Herodotus quoting what someone else said, so a proper title for Herodotus might be the "father of journalism."

According to Herodotus, Thales predicted the eclipse that ended the wars between the Lydians and the Medes. The peace was sealed by a marriage between Astyages, the King of the Medes, and Aryenis, the sister of Croesus, the King of the Lydians. Croesus then turned his attention south, towards the Greeks, so by Thales's time, most of Ionia had been absorbed by Lydia. Miletus, recognizing the futility of resistance, allied itself with Lydia and retained a measure of independence.

Part of the secret of the success of the Lydian Empire was a remarkable invention made around 650 B.C.: coinage. The value of money is often underrated and frequently misunderstood. Economic activity relies on trade, but trade requires two people, each having what the other person wants. Money serves as a universally desirable commodity. The problem is making theory and practice coincide. One way is to make money out of silver or gold, both rare, durable metals.

However, silver or gold by themselves are merely valuable; they are not money. Imagine the problems that would arise from trying to pay for a purchase using silver or gold. First,

[1] Greek geometers used the term "equal" to mean one of three things. First, two figures are equal if one can be superimposed on the other through a sequence of rigid transformations; today we use the term "congruent." Alternatively, two figures were equal if one could be dissected and reconstituted to form the other; today we would say the two figures have the same area. Finally, two figures were equal if their lengths, areas, or volumes had a ratio of one to one.

the purchaser would have to cut off a sliver of metal and measure its size (weighing would be the easiest method). But before accepting it the prudent seller would verify its weight and purity. Discrepancies over perceived weight or purity would lead to endless argument, and trade would grind to a halt. In the seventh century B.C., Croesus's predecessors had a marvelous idea: a trusted authority could stamp a disk of metal guaranteeing its weight and purity. Thus, coinage was invented. The commerce flowing through Lydia made her kings immensely wealthy, so the Greeks used "rich as Croesus" to describe anyone of great wealth.

Around 553 B.C., a usurper named Kurush from Fars rebelled against Astyages, and deposed him by 550 B.C. in a relatively bloodless civil war. Kurush, his homeland Fars (east of Mesopotamia), and his family, the Hakhamani, are better known to us by the Latinization of the Greek forms of their names: Cyrus, Persia, and the Achaemenids. Croesus saw an opportunity to invade and conquer the Medes (which he could do under the guise of liberating them from a foreign conqueror). Before invading, he consulted his advisors, and got an outside opinion from the Oracle of Delphi, supposedly possessed of the gift of prophecy. She told him that if he invaded, he would bring down a mighty empire. This sounded like good news to Croesus, and he sent in his army.

The border of the Lydian Empire was the river Halys. Herodotus noted (with doubt) a belief among the Greeks that Thales diverted the river so a bridge could be built across it. If this story is true, then Thales, like so many later mathematicians, was a military engineer. Croesus's army invaded Persia, and promptly fulfilled the Oracle's prophecy. Unfortunately the mighty empire that Croesus brought down was his own, which Cyrus conquered in 547 B.C. Ionia, as part of the Lydian Empire, became part of the Persian Empire. Again, Miletus remained independent.

Cyrus employed a novel strategy: tolerance. For example, the worship of Marduk had been suppressed by the Chaldean king Nabonidus; Cyrus promised to restore the worship of Marduk. With the worshippers and priests of Marduk on his side, Cyrus conquered the Chaldean Empire easily in 538 B.C. After the conquest, Cyrus kept his promise.

Another group disaffected by the Chaldeans were the Jews, many of whom had been living in Babylon since the time of Nebuchadrezzar. Cyrus allowed the Jews to return to Palestine to rebuild the temple. Few bothered: most had made comfortable lives for themselves in Babylon, though they did send monetary gifts with the returnees.

While in Babylon, the Jews maintained their cultural identity by a rigid adherence to the old customs. The Palestinian Jews, on the other hand, let their customs evolve naturally. As a result, the Babylonian Jews, returning to Palestine, felt the indigenous Jews were practicing the "wrong" form of Judaism. Since the chief city of the Palestinian Jews was Samaria, this was the beginning of a great schism in Judaism and as a result, the Babylonian Jews (who became dominant) learned to hate and detest the Samaritans. Half a millennium later, this hatred would be used to good effect in a story that argued for the fundamental unity of all mankind.

Egypt was added to the Persian Empire in 525 B.C. by Cyrus's son Cambyses, and by the time of his death in 521 B.C., the Persian Empire included Egypt and most of the Fertile Crescent. It was the largest empire the West had yet seen.

During the conquest of Egypt, a number of Greeks were captured by the Persians and transported to Babylon. There is a tradition that PYTHAGORAS of Samos (580–500 B.C.)

was among this group. Pythagoras had been in Egypt since 535 B.C., and spent five years in Babylon before making his way back to Greece. In 518 B.C., Pythagoras left the eastern Mediterranean forever and settled in Croton, at the heel of the "boot" of modern Italy.

Croton was the home of Milon (or Milo) the wrestler, the most famous athlete in antiquity. Milon's strength was legendary. During an Olympic procession around 540 B.C., he is said to have carried a fully grown ox across the stadium, a distance of about 600 feet. Milon, as the local sports hero, was greatly honored in Croton and in 510 B.C. led an expedition that destroyed the neighboring town of Sybaris, whose inhabitants were well known for their rich lifestyle (hence the word *sybaritic*).

At Croton, Pythagoras established a secretive, mystical school that lasted about a century. At least one story suggests that Milon was Pythagoras's patron, and that Milon's daughter became one of the first Pythagoreans. None of Pythagoras's own work has survived; we have only what his followers claim he said. Some of the rules of the school, such as always wearing white and not eating beans, have parallels with ancient Egyptian practices. Pythagoras and his followers were struck by the many relationships among numbers; they sought to analyze the physical world in terms of these number relations. For example, they apparently discovered that the sum of the first n odd numbers is n^2. Since the Greek word for number is *arithmos*, this study of number properties became known as arithmetic.

Perhaps the best evidence of the beauty of mathematics comes from the Pythagorean study of music. According to one rather dubious story, Pythagoras happened to be passing by a smithy, and noticed that the noise of the falling hammers sounded pleasant. Upon investigation, he discovered that the weights of the hammers had a whole number ratio to one another. This began a study of music from a mathematical perspective.

Rather than using hammers, the Pythagoreans used a monochord: a one stringed instrument with a movable bridge. If the bridge divided the string into two equal parts, the two parts could be plucked one after another (melodically) or simultaneously (harmonically). In both cases, the sounds went together well: the string ratio of 1 to 1 produced a consonance (now known as unison, from the Latin "one sound").

Suppose the bridge was moved so the string was divided in a 2 to 1 ratio. The shorter part would produce a higher pitched version of the tone produced by the longer part, and the $2:1$ ratio produces another consonance.[2]

Another consonant ratio corresponds to string lengths in a ratio of $3:2$. Finally, an "inversion" takes the lower note of a set and replaces it with a higher pitched version of the same note (in this case, by halving the string length). Inverting the $3:2$ ratio produces a $2:3/2$ ratio, which we can simplify to $4:3$. The evidence of the senses, as well as the elegance of the numerical ratios, make us regard this ratio as consonant. Moreover, the three lengths together give us a $6:4:3$ ratio, which has a remarkable property: the ratio of the difference between the first and second to the difference between the second and third is equal to the ratio between the first and third. In this case, the ratio of $6-4$ to $4-3$ is equal to the ratio of 6 to 3. This type of ratio is now called a harmonic ratio (and the numbers are said to form a harmonic progression). On the other hand, if we began with the $4:3$ ratio and inerted it, we would obtain a $3:2$ ratio, giving us three numbers in an arithmetic progression.

[2]Changing the tension or thickness of the string also affects the tone. This fact makes the hammer story extremely improbable, since several factors would influence the fundamental tones produced by dropping hammers.

Suppose we tuned an instrument so the strings had a $4 : 3 : 2$ or a $6 : 4 : 3$ ratio. The instrument would produce three notes, with the property that any combination of the notes would form a consonance. However, a three note repertoire is rather limited, so additional notes were added. Various schemes were tried, but eventually the division of the $2 : 1$ interval into eight notes became standardized (Euclid was the first to note this division, though it certainly predated him). Consequently, the $2 : 1$ ratio is said to correspond to an interval of an octave. From lowest to highest, these notes are now designated as C-D-E-F-G-A-B, with the eighth note also called C and beginning the pattern again.[3] The C-G interval, which spans five notes, corresponds to the $3 : 2$ ratio; hence this ratio is referred to as a fifth. Likewise the C-F ratio, which spans four notes, corresponds to the $4 : 3$ ratio, and is designated a fourth. This means that the F-G interval must correspond to a ratio of $9 : 8$. We can designate this ratio as defining a tone. But the internote ratios cannot all be $9 : 8$, since $9^7 : 8^7 \neq 2 : 1$. In fact, the problem is worse than that: there are *no* whole numbers p, q for which $p^7 : q^7 = 2 : 1$. Thus it is impossible to divide the octave into equal intervals. This is the root of what is called the tuning problem: how can we make a product of the power of one rational number equal another rational number? If the two rational numbers have different prime factors, then the problem is unsolvable; the best we can hope for is an approximation.

Any solution to the tuning problem must sacrifice some of the intervals. In Pythagorean tuning, the octave and fifths are retained, while the fourths and tones are sacrificed where necessary. For example, if C-D and D-E both correspond to a $9 : 8$ ratio, then to make C-F correspond to a fourth (and thus the ratio $4 : 3$), then E-F must correspond to the ratio $256 : 243$. Since $256^2 : 243^2$ is approximately equal to the ratio $9 : 8$ that defines a tone, this new and inelegant ratio is designated a semitone. In order for all fourths to be perfect, it is necessary that any sequence of four notes consist of two tones and a semitone. Likewise, for all fifths to be perfect, any sequence of five notes must contain three tones and a semitone. The octave, which consists of a fourth and a fifth together, thus consists of five tones and two semitones, with the semitones five notes apart. Reconciling all these factors produces the Pythagorean tuning:

C to D	D to E	E to F	F to G	G to A	A to B	B to c
9 : 8	9 : 8	256 : 243	9 : 8	9 : 8	9 : 8	256 : 243

The F-B fourth is slightly sharp (the upper note is slightly high in pitch), but the remaining fourths and fifths are perfect. This solution is uniquely determined up to the starting point. In the above, the lowest note of the scale is C (so we have two tones followed by a semitone, followed by three tones and a semitone), but if the lowest note is E (and thus we begin with a semitone, followed by three tones and a semitone, then two tones), you are playing in the "Dorian mode."

In addition to studying music, Pythagoras began a tradition of examining mathematical results in an "immaterial" and "intellectual" manner, thereby turning it into a "liberal art." This probably means that Pythagoras introduced the deductive method to mathematics, while restricting its domain to the theoretical properties of abstract objects. We can

[3]To distinguish the two Cs, the German physicist Hermann von Helmholtz used a system of marks. Thus if our first note is C, then the same note one octave higher is designated c, and the same note one octave higher still is designated c′. This last note corresponds to "middle C."

draw an interesting parallel: the ancient Olympic games highlighted the skills of the warrior, presented in an abstract form divorced from an actual siege or battle. Although the term itself is Latin, the idea of liberal arts originated with the Greeks—and slavery. Slavery was a key element of almost every culture before the present day, though generally men (and women, and children) were slaves because of military conquest, rather than race. Pythagoras himself was probably a slave during his time in Babylon.

Slaves worked with their hands: hence, the work done by slaves came to be known as manual labor, from *manus* ("hand" in Latin). Free men, on the other hand, were expected to pursue the liberal arts, from the Latin *libera* ("free man" in Latin). Free women, incidentally, were expected to bear children; *raising* the children was a task for slaves.

The mathematics of the Egyptians and the Babylonians concerned itself with practical matters, whether it was computing the height of a pyramid or the cost of digging a canal. Because of its association with manual labor, this type of mathematics was deemed fit only for slaves. The Greeks gave the name logistics to this practical, computational mathematics. The difference between logistics and a liberal art like geometry can be illustrated in the following way. The procedure for finding the area of a parallelogram:

Rule 2.1. *The area of a parallelogram is the base times the height.*

is logistics, while the theorem:

Theorem 2.1. *If two parallelograms have equal bases, and equal heights, then either can be dissected and rearranged to form the other.*

is geometry.

The Pythagoreans seem to have been the first to make proof an essential part of mathematics. They are credited with discovering and proving at least five theorems. Four of these are:

1. The angles in a triangle are together equal to two right angles.
2. The angles in an n-gon are together equal to the angles in $n - 2$ triangles.
3. The exterior angles in a polygon are together equal to four right angles.
4. The space about a point can be filled with regular triangles, squares, or hexagons.

The last may have led to Pythagorean discovery of three of the five regular solids: the tetrahedron, formed by equilateral triangles; the cube, formed by squares; and the dodecahedron, formed by regular pentagons.

The fifth Pythagorean discovery was the Pythagorean Theorem. A few myths about the Pythagorean Theorem ought to be discussed. The most often repeated myth comes to us from Proclus (writing in the fifth century A.D.): "Those who like to record antiquities" claim Pythagoras sacrificed an ox (a hundred oxen in some accounts) upon the discovery of the relationship between the sides of a right triangle. Proclus sounds dubious: the Pythagoreans believed in transmigration of souls, and were adamantly opposed to animal sacrifices. The story is also suspiciously like one told about Thales. Another story is that Pythagoras learned the theorem in Egypt, from "rope stretchers" who routinely formed 3-4-5 right triangles using a knotted chord. There is no evidence of any Egyptian knowledge of the Pythagorean Theorem in any form.

If Pythagoras did not discover the theorem independently, then he may have learned about it in Babylon. More concretely, Pythagoras may have learned how to construct what are now called Pythagorean triplets: three numbers, a, b, and c, that satisfy $a^2 + b^2 = c^2$. Pythagoras constructed triplets in the following way: if a is any odd number, b half of one less than the square of a, and c one more than b, then $a^2 + b^2 = c^2$. For example, if $a = 5$, $b = \frac{1}{2}\left(5^2 - 1\right) = 12$, and $c = 12 + 1 = 13$, and $5^2 + 12^2 = 13^2$.

Both the Pythagorean Theorem and the tuning problem lead to incommensurable quantities, discovered by the Pythagoreans during the fifth century B.C. Today we would say that two quantities are incommensurable if the ratio between them corresponds to an irrational number. We do not know how incommensurable quantities were discovered or who discovered them, and even the identity of the first pair of incommensurable quantities is unknown. One of the better candidates is the side and diagonal of a regular pentagon inscribed in a circle. This suggests that the discoverer was HIPPASUS (fl. ca. 430 B.C.) from Metapontum, who was apparently expelled from the order for revealing to outsiders the Pythagorean methods of inscribing a regular pentagon in a circle and a regular dodecahedron in a sphere.

The school did not long survive the discovery of incommensurable quantities, for it began to interfere in local politics. By the middle of the fifth century B.C., it was suppressed by the authorities. Real progress in mathematics came from the Greek heartland—which was at that point in a struggle for its very existence.

2.1.2 The Wars of Greece

The greatest of the Persian kings was Cambyses's successor Darius, from a collateral branch of the Achamaenids. Darius became king in 521 B.C., and concentrated on internal improvements. He established a vast system of roads, complete with military patrols to deter bandits, and an efficient postal system. Regarding the latter, Herodotus wrote: "Neither rain, nor sleet, nor dark of night stays [prevents] these couriers from the swift completion of their appointed rounds."[4]

In 500 B.C. the Ionian city-states, led by independent Miletus, revolted against Persian rule. They pled for help from the Greek mainland, but only Athens and Eretria sent more than token assistance. Miletus finally lost its independence when the Persians conquered it in 494 B.C., and Darius organized a punitive expedition to deal with the interfering Greeks. Darius, a shrewd diplomat, encouraged the neutrality of the other Greek city-states by announcing that his battle was with Athens and Eretria alone.

In 490 B.C., twenty thousand Persian troops laid siege to Eretria. After six days, a traitor opened the city gates to the Persians, who destroyed the city and carried its population off as slaves. With Eretria successfully reduced, the Persians crossed the narrow strait separating Euboea from mainland Greece, and made ready to crush Athens. The Athenians sent a runner, Pheidippides, to seek help from Sparta.

The Spartans conquered their neighbors the Messinians around 600 B.C., and reduced the inhabitants to the status of serfs, or *helots*. The helots outnumbered the Spartans ten to one, so to keep the helots subjugated, the Spartans made their society increasingly militaristic. Children were subject to a rigorous examination upon birth, and any who seemed

[4] The architect William Mitchell Kendall thought the phrase appropriate for the New York General Post Office, and when the building was completed in 1912, the inscription appeared along its façade. It is not the motto of the U.S. Postal Service.

weak or sickly were exposed and left to die. Education emphasized physical fitness, poetry, and music.

At the age of 20, a Spartan male joined one of several military units. The members of the unit ate together, lived together, fought together—and if necessary, died together. One Greek visitor, after eating at the communal barracks, found the food so unpalatable that he is said to have remarked, "Now I know why the Spartans are unafraid of death." Silence was encouraged, and if one had to speak, one should be brief about it and issue short statements that cut right to the point. Since Sparta is located in the region of the Peloponnesus known as Laconia, the other Greeks commented on their *laconic* way of speaking.

The Spartans agreed to help Athens, but for political reasons, they could not send assistance until after the full Moon. Pheidippides ran back to Athens to report the bad news. Boldly, the Athenian commanders made the decision to attack before the Spartans arrived. This turned out to be the correct decision, for the lightly armed Persians were no match for the heavily armed and armored Greek hoplite (which referred to their armor). The Persians were forced back to their ships, losing over six thousand men in the process. The Athenians lost less than two hundred. Pheidippides ran the 26 miles from the battle site to the marketplace in Athens, where he announced victory (*nike* in Greek) over the Persians on the beaches of Marathon. Then he collapsed and died of exhaustion, having run about 150 miles in two days.

Egypt revolted against the Persians at the same time, and keeping the rich province of Egypt was far more important than punishing the Greeks. Thus the next great invasion of Greece did not occur until 480 B.C., under Darius's successor Xerxes. This time it was clear that the intent was not to punish Athens, but instead to conquer all of Greece.

At the pass of Thermopylae, 1400 Greeks, including 300 Spartans, faced the entire Persian army. The exact size of the Persian army is unknown, but it may have been around 100,000 soldiers, which included 10,000 elite troops, known as "Immortals." According to Herodotus, the Spartans were warned by a native of Trachis (a village near Thermopylae) that when the Persians fired their arrows, they would be so numerous they would blot out the sun. A Spartan named Dienices remarked, "Good! We will have shade to fight in." Defeat was inevitable, but the Spartans stopped the Persian advance for three days (dying to the last man), buying enough time for other parts of Greece to improvise hasty defenses. It was not enough: Xerxes laid siege to Athens and burned it to the ground. But at the naval battle of Salamis, the Persians lost nearly half their fleet; Greek losses were insignificant. Xerxes withdrew—for one year. The next year, Xerxes returned with another army, suffered another defeat (at Plataea), and withdrew again.

To fight Persia, Athens organized an alliance, now known as the Delian League, because its headquarters and the treasury were located on the island of Delos. Member states of the alliance could either contribute ships and men, or the equivalent in money. Most members chose to supply money, letting Athens build and man the ships of the fleet. Eventually the Delian League included all of Ionia, and the Persian threat began to recede.

In theory the alliance, with no enemy to fight, ought to have been disbanded, but Athens argued that it was still necessary to defend Greece from Persia. Member states grudgingly agreed to maintain what was rapidly becoming the Athenian Navy, but after a while, the threat of Persian invasion seemed remote. Tired of paying for a navy that did it no good, the inhabitants of the island of Naxos attempted to withdraw from the alliance. The Athenians

Figure 2.1. The Athenian Empire

refused to accept the withdrawal, laid siege to Naxos, and in 470 B.C., captured the city, destroyed its fortifications, and sold the inhabitants into slavery. Other member states were cowed into submission, their annual dues being converted into tribute, and the Athenian Empire was established.

Pericles was an ambitious aristocrat who came to power in 461 B.C. and presided over the Golden Age of Athens. The great playwrights Aeschylus and Sophocles lived and worked in Athens in this period; indeed, one of Pericles's first recorded actions was paying for the production of one of Aeschylus's plays in 472 B.C. The Parthenon, considered one of the most elegant buildings in the world, was also constructed in the time of Pericles, using funds technically belonging to the Delian League. When Thucydides of Melesias objected to this misappropriation of funds, Pericles arranged to have him ostracized (a political maneuver that forced Thucydides into exile for ten years).[5]

The Golden Age of Athens was a great era of building, democracy, literature—and slavery. At the height of the Golden Age, there were perhaps fifty thousand (male) citizens with the franchise—and over a hundred thousand slaves. Many worked at the state-owned silver mines of Laurion, which were so profitable that by 483 B.C. each Athenian citizen received a share of the revenues. Even today, mining is an extremely dangerous profession; in the fifth century B.C., the death toll among the slaves must have been horrific, and some estimate that slaves only survived an average of two years in the mines.

Only three states of the Delian League maintained a measure of independence, contributing ships and men to the fleet rather than money to the treasury. Geography may have played a role: all were large islands off the shore of Asia Minor, so of necessity they were forced to have strong navies. From north to the south, the islands were Lesbos, Chios, and Samos.

[5]The term comes from *ostraka* ("pottery shard") onto which the name of an individual might be written: those who received more than a specified percentage of the votes were sent into exile, with the understanding that their property and person were to remain undisturbed in their absence.

2.1.3 Mathematicians of the Golden Age

Chios joined the league in 479 B.C. There may have been a school there established by Pythagoras, for OENOPIDES of Chios (fl. 450 B.C.) was one of the more celebrated mathematicians of the era. We know nothing about Oenopides's life, though there is some evidence he visited Athens. He would not have been the first: as the vibrant center of a growing empire, Athens attracted the best minds of the age. For example, Herodotus moved to Athens to write his *Histories* about the recent Persian wars. Oenopides is credited with being the first to construct a perpendicular and to construct an angle equal to a given angle. As the Parthenon and other large buildings were under construction by the time of Oenopides, it seems unlikely that he was actually the first to do either of these things. More likely, Oenopides was the first to construct the figures using *only* compass and straightedge, the tools of the liberal arts, not those of the manual ones.

Around 462 B.C., ANAXAGORAS (500–428 B.C.) came to Athens from Clazomenae, in Ionia. Anaxagoras may have been the first to bring the Ionian tradition of rational inquiry to Athens. Anaxagoras suggested that the sun was not a god, but instead a hot rock larger than the Peloponnese. This suggestion might have been based on the following logic: First, the Moon eclipses the sun, which implies the sun is more distant. Second, the sun and Moon appear to be the same size, so the sun must be larger than the Moon. Hence the region of totality during an eclipse must be smaller than the Moon. From accounts of the total eclipse of April 30, 463 B.C., Anaxagoras might have learned that the shadow of the Moon covered most of the Peloponnese; this would allow him to give a lower bound for the size of the Moon and sun.

The other Athenians were not as impressed with the Ionian traditions, and Anaxagoras was charged with heresy. In an account written five hundred years later, Plutarch said Anaxagoras "wrote on" the squaring of the circle while awaiting trial, though Plutarch gave no details. The events surrounding the trial itself are somewhat hazy; Anaxagoras was apparently condemned to death, though in fact he lived out the remaining years of his life in Lampsacus in Ionia. Pericles may have secured Anaxagoras's acquittal (there is a tradition that Pericles was a student of Anaxagoras), but other accounts suggest that Anaxagoras had already left Athens by the time charges were brought, and that the trial and condemnation occurred *in absentia*.

If Plutarch's account can be taken at face value, Anaxagoras may have been the first to examine one of what became known as the Three Classical Problems of antiquity:

1. **Trisection of an angle**: given an angle, divide it into three equal angles.
2. **Duplication of the cube**: given a cube, to construct another with twice the volume.
3. **Squaring the circle**: given a circle, to construct a square equal to it in area.

The greatest mathematician of the era was Oenopides's countryman, HIPPOCRATES (470–410 B.C.). Hippocrates of *Chios* should not be confused with his contemporary, Hippocrates of *Cos*, the physician. Hippocrates of Chios was originally a merchant who came to Athens to retrieve a cargo lost to piracy. The legal proceedings against the pirates took so long that, to support himself, Hippocrates became the first known professional teacher of mathematics. Moreover, he wrote a textbook, called the *Elements of Geometry* (now lost), for his students.

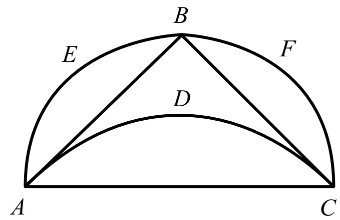


Figure 2.2. Hippocrates Lunes

Hippocrates was apparently aware that:

Theorem 2.2. *The area of a circle is proportional to the area of the square on its diameter.*

However, he probably did not have a proof.

While attempting to square the circle, Hippocrates was able to show that certain lunes (regions bound by circular arcs) were equal in area to certain rectilineal figures (see Figure 2.2). If ABC is an isosceles right triangle inscribed in a semicircle, and ADC is a segment of a circle similar to the segments AEB, BFC (two segments are similar when their central angles are equal), then the lune $ABCD$ is equal to the triangle ABC. Hippocrates found several other results, all equating the area of a lune with the area of a certain rectilineal figure.

Hippocrates also took steps towards duplicating the cube. Given two quantities, a and d, then the problem of inserting one mean proportional between them is the problem of finding b so $a : b = b : d$. We could insert two mean proportionals, b and c, if $a : b = b : c = c : d$. Hippocrates showed that in this case, $a^3 : b^3 = a : d$. Thus if $d = 2a$, then the cube with a side of b will have twice the volume of a cube with a side of a. Hippocrates himself was unable to solve the problem of finding two mean proportionals.

The most enigmatic mathematician of the era was DEMOCRITUS (ca. 460–370 B.C.), who came from Abdera at the northern edge of the Athenian Empire. Democritus traveled around the Mediterranean, and made at least one visit to Athens, where he was snubbed by Anaxagoras. Although none of Democritus's writings have survived, references to his work in other sources indicate that he was the first to give correct formulas for the volume of a pyramid and cone, though he apparently did not give a proof. A work of his known only by its title, *Two Books on Irrational Lines and Solids*, suggests that the existence of incommensurable figures was well known by his time.

HIPPIAS (b. ca. 460 B.C.) also came to Athens, from Elis, an independent city-state on the western shore of the Peloponnesus. He was a member of a new school of philosophy, the Sophists. Unlike the Pythagoreans, who were secretive and would teach only those who wished to become Pythagoreans, the Sophists would teach anyone who paid them. Teaching seemed too much like manual labor to other philosophers, who reviled the Sophists. It might seem that there was a small market for philosophy lessons, but one of the skills taught by the Sophists was the skill of debate, critical for success in the Athenian legal system.

Hippias might have taught in Sparta briefly, but found the Spartans uninterested in his primary subjects (astronomy, geometry, and logistics). Thus he made his way to Athens, where his skills were in demand. He was the first to invent a curve that could *not* be constructed using compass and straightedge. Since the curve could be used to square the circle,

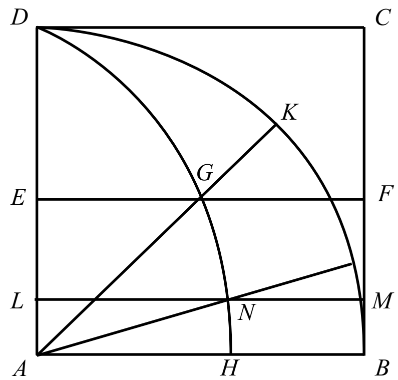


Figure 2.3. Trisectrix or Quadratrix

it is known as the quadratrix, from the Latin *quadratus*, "square" (see Figure 2.3); it is not known if Hippias knew that the curve could be used in this fashion.

To construct the quadratrix, take the square $ABCD$, and let DKB be the quadrant of a circle. Let the side DC drop parallel to itself towards AB while at the same time the radius AD rotates until it coincides with AB. The intersection of the radius and the side of the square at points G, N, etc. form the quadratrix. If angle KAB is to be trisected, then draw EF parallel to AB and divide EA into thirds at L (which can be done using compass and straightedge); draw ML which intersects the curve at N. Then $\angle NAB$ is one-third $\angle KAB$. Squaring the circle relies on the theorem that arc DB is to DA as DA is to AH; thus locating the point H on the quadratrix allows us to square the circle.

ANTIPHON (480 B.C.–411 B.C.) and BRYSON (b. ca. 450 B.C.) were two other important Sophists. Antiphon was Athenian, and Bryson may have come to Athens to study under Socrates. Antiphon worked on the problem of squaring the circle, and apparently suggested that its area could be found by considering the area of an inscribed polygon; this suggests an early use of the method of exhaustion. Bryson took Antiphon's method further and bounded the area between the area of inscribed and circumscribed polygons.

The problem of squaring the circle attracted enough popular attention by this time to warrant a reference in *The Birds* (414 B.C.) by Aristophanes. The main characters, Euelpides and Pisthetairos, leave Athens to found the utopian "Cloud Cuckoo Land," and are subsequently bombarded by unsolicited and impractical advice. The astronomer METON (b. ca. 440 B.C.), whose observation that 235 lunar months very nearly equals 19 solar years led to the development of a very accurate luni-solar calendar, appears as a character with a plan to design the city: "With this straight ruler here I measure this, so that your circle here becomes a square." Pisthetairos replies, "This man's a Thales," then drives him away with a beating.

Beating geometers played only a minor role in Aristophanes's work. A more constant theme was the stupidity and pointlessness of war. Athens's quest to build an empire led to the Great Peloponnesian War, which began in 431 B.C. In response, Aristophanes wrote *The Acharnians* (425 B.C.), *The Knights* (424 B.C.), and *The Peace* (421 B.C.), criticizing the leadership and conduct of the war, and calling for its end. The plays were well-received, but war proved more popular. The Athenians also passed a law limiting political satire, though this did not prevent Aristophanes from writing his greatest work, *Lysistrata* (411 B.C.). In it, Lysistrata convinces the wives and mistresses of the Athenian and Spartan soldiers to withhold sex until they cease their endless war.

Lysistrata ends happily. The Great Peloponnesian War ended in 404 B.C. with the destruction of the Athenian Empire. Sparta emerged the dominant power in the Greek world. Athens, having lost her worldly empire, was about to establish one far greater, and far more important. Built on ideas, not force, it would prove far more lasting, and to this day we are very much a part of the Athenian intellectual empire.

2.1.4 Mathematicians and the Academy

Plato, a student of Socrates, fought in the last five years of the Peloponnesian War, but was born to an aristocratic family and had political ambitions. But in 399 B.C. Socrates was ordered to commit suicide on charges of having "corrupted youth." This convinced Plato that politics was no place for a man with a conscience, and he left Athens. Plato spent the next dozen years traveling about the Mediterranean. He visited Egypt and southern Italy, where he met Pythagoreans and taught Dion, the brother-in-law of the Tyrant of Syracuse, Dionysius.[6]

According to one story, Dionysius grew angry at Plato and arranged to have the philosopher sold into slavery. Plato was saved from this unhappy fate by ARCHYTAS (428–347 B.C.), the leading citizen of Taras. Taras (Tarentum in Latin, and now Taranto, Italy) was the only colony ever successfully founded by Sparta, populated by bastard children born during the Messinian Wars. Spartan culture valued military prowess; Archytas had been chosen as *strategos* (general) seven times, and was never defeated.[7] Spartan culture also valued music; Archytas is believed to have invented the $\pi\lambda\alpha\tau\alpha\gamma\eta$, which Aristotle derided as a rattle, useful for keeping children occupied so they do not break things (although it has been suggested that the $\pi\lambda\alpha\tau\alpha\gamma\eta$ is the same instrument that adorns a number of vases of the fourth century B.C. alongside a number of other religious symbols). Archytas is also credited with associating the arithmetic, geometric, and harmonic means with musical intervals, and was the first to solve the problem of duplicating the cube, using a complex method involving the intersection of two space curves.

When Plato returned to Athens in 387 B.C., he founded the most famous school in history: the Academy. Plato may have been the first to use the term "mathematics", which comes from the Greek *mathema* ("that which is learned"). He emphasized mathematics as a way to train the mind in deductive thinking, so the Academy became a center for mathematical teaching and research. Archytas joined the faculty almost immediately.

The name of the school commemorates a rather unsavory event in the history of Athens. According to one story Theseus, a king of Athens and great hero, abducted Helen, a princess of Sparta (later to become Helen of Troy). An Athenian named Academus revealed where Theseus hid Helen, and she was rescued by her brothers Castor and Polydeuces (Latinized as Pollux). Tradition placed Academus's estate just outside the city walls of Athens; the site was purchased by a wealthy admirer of Plato and donated to him.

[6] In Greek political theory, a "tyrant" is the term used for any non-hereditary ruler.

[7] It is not clear which campaigns he fought in, but southern Italy was a constant battleground during the time period, and since, by tradition, a *strategos* could not succeed himself, the continued election of Archytas had to have some basis.

Other stories are told about the school. Johannes Tzetzes, a Byzantine author, claimed that a plaque above its entrance read, "Let No One Unversed in Geometry Come Under My Roof." Since Tzetzes wrote six hundred years after the Academy closed its doors forever, the claim is dubious. Another story highlights the difference between the manual and the liberal arts: a student asked Plato the value of knowledge, at which point Plato told a servant to give the student a coin, "since he must have value for what he learns." Then the student was expelled from the Academy.

The greatest mathematician associated with the Academy was EUDOXUS of Cnidus (408–355 B.C.), a student of Archytas in Tarentum. The life of Eudoxus coincided with a resurgence of Persian interest in Greece. Athens, Thebes, Corinth, and Argo all had historic grudges against Sparta, and the Persians were well aware of this. Late in 396 B.C., Pharnabazus, a *satrap* (provincial governor) of the Persian Empire, let these cities know that if they attacked Sparta, they would receive his support. Thus, less than a century after the Greeks united to fend off the Persian Empire, they united again—this time at the urging of the very same Persian Empire against one of their former allies. In 395 B.C., the Corinthian War began.

A Spartan fleet was destroyed just offshore Cnidus in 394 B.C., when Eudoxus was fourteen. Since the destruction of Sparta would leave expansionist Athens supreme among the Greek city-states, Persian policy reversed itself and in 387 B.C., the Spartans and Persians negotiated the "King's Peace," binding on all of Greece, though no Greek city-state besides Sparta was consulted.

Eudoxus visited Athens shortly afterwards. He was so poor he had to stay at the Piraeus (the port section), and walked ten kilometers uphill each day to the Academy. Eudoxus only stayed in Athens a few months, before going to Egypt. There a sacred bull licked his cloak, which meant (according to the priests) that he would die young, but famous. After a year in Egypt, he went to Cyzicus, where he founded a school before returning to Athens around 368 B.C. While Plato was away in Sicily, attempting to tutor King Dionysius the Younger of Syracuse, he left the Academy in the hands of Eudoxus. Dionysius, like his father, grew angry with the philosopher and eventually Plato returned to Athens. This allowed Eudoxus to return to Cnidus where he stayed the rest of his life.

Eudoxus was responsible for developing the theory of proportion and ratio, a crucial step in the development of the idea of a real number. Two magnitudes were said to have a ratio if either could be multiplied to exceed the other (thus, a circle and a square could have a ratio, but not a circle and a line). To compare two ratios, Eudoxus used the definition:

Definition. *Two ratios are equal if, given any equimultiples of the first and third, and any equimultiples of the second and fourth, the equimultiples of the first and second, and the equimultiples of the third and fourth, alike equal, exceed, or fall short of the latter equimultiples.*

What this complicated definition means is that given any two ratios, $\frac{A}{B}$ and $\frac{C}{D}$, and any whole numbers m and n, then if $mA = nB$ whenever $mC = nD$, and $mA > nB$ whenever $mC > nD$, and $mA < nB$ whenever $mC < nD$, then the ratios are equal. Another way to interpret Eudoxus's complex definition is that if two ratios $\frac{A}{B}$ and $\frac{C}{D}$ are equal, then given any rational number $\frac{n}{m}$, the ratios are both greater than, both less than, or both equal to the rational number.

With the theory of proportions (which Euclid preserved as Book V of the *Elements*), Eudoxus was apparently able to prove the propositions of Democritus and Hippocrates regarding the volumes of cones and pyramids, and the area of a circle. However, we have no copies of his proofs.

MENAECHMUS (fl. 350 B.C.) was a student of Eudoxus in Cyzicus, and was one of the first to study the conic sections in a systematic fashion.[8] He discovered specific properties, or symptoms of the conic sections, that translate into the modern algebraic equations $ky = x^2$ or $kx = y^2$ for the parabola and $xy = k^2$ for a hyperbola. Using these symptoms, Menaechmus presented a new and simple method of duplicating the cube. Hippocrates had shown that this problem reduced to inserting two mean proportionals between two quantities; Menaechmus showed that these two mean proportionals could be found using the intersection of a parabola and a hyperbola. In particular, suppose we wish to insert two mean proportionals between a and b. We seek to find x, y that satisfy the proportionality $a : y = y : x = x : b$. From the first and second terms of the proportionality, we have $a : y = y : x$; hence $ax = y^2$, so x, y, are on a parabola. From the first and third terms, we have $a : y = x : b$; hence $ab = xy$ and x, y are on a hyperbola. The intersection of the parabola $ax = y^2$ and $ab = xy$ can be used to find the desired quantities x and y.

One of the stories told about Menaechmus is that he came into the service of the household of Philip, the king of Macedon as a tutor for the young Prince Alexander. After a particularly difficult lesson, Alexander asked whether there was an easier way to learn mathematics. “Sire,” Menaechmus replied, “there is no royal road to geometry.”

2.1.5 The Hellenistic Kingdoms

By the time of Philip of Macedon, it was obvious that the Persian Empire was nothing but an empty shell, ready to topple at the slightest push. Philip intended to supply that push. After uniting the Macedonian tribes and modernizing their army, he conquered the Greeks and prepared to march against the Persian Empire. Unfortunately he was assassinated in 336 B.C.by a conspiracy that included his ex-wife Olympias.

The conspiracy placed their son Alexander on the throne. Over the next ten years, he embarked on a career of conquest never before seen in the ancient world, conquering an empire that stretched from Greece to the borders of India. He would have gone further, but his soldiers, who had faithfully followed him across the width of what was once the Persian Empire, rebelled at the thought of getting even farther from home. Thus, Alexander turned back. There is a story that he wept at the thought that there were “no more worlds to conquer,” though this is probably apocryphal: Alexander knew that India lay just beyond the borders of his empire.

It cannot be denied that Alexander was a brilliant tactician, but his Persian campaign offered few opportunities to show it. The Persians never really learned how to fight the heavily armed Greek hoplites that defeated them at Marathon and Plataea, so the conquest of the Persian Empire was a relatively simple strategic exercise. Alexander’s true greatness appeared in his domestic policy after the conquest. The empires of the past were conquered by force, maintained by force, and ultimately destroyed by force. Alexander intended to break the pattern.

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[8]It may be significant that neither Eudoxus nor Menaechmus learned from their teachers while they were *at* the Academy.

First, he had to foster unity among his subjects. One way to do so was to give them a common heritage. In the ordinary course of events, the civilization of the conqueror inevitably percolated downward to the conquered, but Alexander hoped to expedite the process by establishing Greek colonies throughout the empire. According to his biographer, he founded seventy towns, though many of these were built around existing cities. At Issus, where he defeated the Persians, he established a new city and named it Alexandria (now Iskenderun, Turkey). At the mouth of the Nile in Egypt, he established another city and named it Alexandria (now al-Iskandarīyah, Egypt). At the eastern edge of the empire, he established yet another city and named it Alexandria (now Kandahar, Afghanistan).

By itself such a policy invited disaster, for the colonists would be viewed as foreign conquerors and, of course, the very name of the city would be a constant reminder. Thus Alexander encouraged his subjects to intermarry with the native population so that, in time, there would be no distinction between the Greek and non-Greek members of the empire. At the ancient Elamite capital of Susa, Alexander and eighty of his officers took Persian wives, and for the roughly ten thousand common soldiers who took Persian brides, he gave generous dowries. He even located the capital of his empire at Babylon, a symbolic choice for it was midway between Persia and Greece. Finally, he recruited Persians for key positions in the government and army.

All his plans for an empire unified by common interest and common heritage came to nothing when he died on June 11, 323 B.C. (probably from malaria, exacerbated by excessive drinking). His generals, who had followed him from Greece to the borders of India and back again, clustered around him, demanding to know who would inherit his empire. Who, they demanded, would be Alexander's heir? Who would the empire go to? They leaned close and heard his reply: "To the strongest." Thus for nearly twenty years, Alexander's generals fought for control of the empire during the Wars of the Diadochoi ("successors" in Greek).[9]

By 305 B.C., the surviving generals made an uneasy truce that divided the empire into five Hellenistic kingdoms. The two most important were those ruled by Seleucus and Ptolemy. The Seleucid Empire included most of what was once the Persian Empire, with the exception of the satrapy of Egypt; Seleucus himself was the only one of Alexander's officers to keep the Persian wife he married at Susa. Egypt was the domain of Ptolemy, and Ptolemaic Egypt would be the longest-lived of the successor states.

Ptolemy established his capital at Alexandria at the mouth of the Nile. There he established one of the most famous research institutions in the world: the Museum of Alexandria, named after a statue dedicated to the nine Muses (the patron goddesses of poetry, dance, music, history, and astronomy) that dominated its entrance. The Museum combined the teaching and research aspects of a university with the displays we associate with modern museums and zoos. It also contained a library which, in time, came to be the best-known part of the museum, and received funds directly from the king, making it one of the first state-supported research institutes.

A goal of the library was to collect every written Greek work: every tragedy, every comedy, every treatise on science and on mathematics. To augment its collection, royal decree

[9] There is an intriguing suggestion: One of Alexander's most honored generals was named Craterus, and in Greek, "To Craterus" is *kraTERoi*, while "To the strong*er*" is *KRATeroi*. Craterus was away at the time, building a fleet in Cilicia.

ordained that ships entering the harbor at Alexandria had to surrender their papyrus to the museum, which would make copies—and the *copies* would be returned to their owners. As a result of this, Alexandria became the publishing center of the Mediterranean. At its peak, the Library contained some 300, 000 Greek manuscripts, while another 200, 000 were kept at the nearby Temple of Sarapis, converted into a library annex around 235 B.C.

Ptolemy was more than a mere patron of the arts: he was himself interested in learning. The following suspiciously familiar story is told: one day, while taking lessons in geometry, he came across a very difficult proposition. Unable to understand its proof, he asked his teacher whether there was an easier way to learn mathematics. "Sire," the teacher replied, "there is no royal road to mathematics." The teacher was supposedly EUCLID (fl. 300 B.C.).

In Euclid's lifetime, Alexandria became the busy port of a prosperous Empire, and Euclid himself may have watched the construction of a new lighthouse (completed around 280 B.C.) on the island of Pharos, just offshore and perhaps a mile from the library itself. Teams of donkeys ascended to the top, bringing wood to burn in a fire that could be seen twenty miles out to sea; the lighthouse became one of the Seven Wonders of the World.

We know almost nothing about Euclid's life. He may have taught for a while at the Museum, and then may have established his own school nearby. An apocryphal story (again, suspiciously familiar) is that a student asked the value of a theorem; Euclid is said to have given the student a coin and dismissed him, chiding him for requiring that knowledge have value.

At the library, Euclid had access to every important mathematical work written by his predecessors. Euclid was not a brilliant mathematician: he discovered no great theorems, created no new methods. What he did create was a vast system for the elementary geometry of the time: a way that theorems could be deduced in a straightforward and logical manner, beginning with ten simple assumptions. Five of these relate to geometrical objects:

1. A straight line may be drawn between any two points.
2. A straight line may be extended in a straight line.
3. About any point a circle may be drawn with any given radius.
4. All right angles are equal.
5. If a line falling on two lines makes the interior angles on one side of the line less than two right angles, the two lines, if extended, will meet on that side.

The last is the so-called parallel postulate.

Notice that the first three postulates assume the existence of straight lines of arbitrary lengths and positions, and circles of arbitrary centers and radii: thus, Euclid's geometry is the geometry of straight lines and circles, and the figures that can be derived from them.

The five "common notions" that relate to logical deduction are:

1. Things equal to the same thing are equal to each other.
2. If equals are added to equals, the results are equal.
3. If equals are subtracted from equals, the results are equal.

4. Things which coincide are equal.

5. The whole is greater than any part of it.

From these ten simple assumptions, Euclid derived over a thousand theorems, which he divided into the thirteen "books" (equivalent to a modern day chapter) of the *Elements*. For the next two thousand years, the *Elements* dominated mathematics: one was not a mathematician, but a *geometer*. Indeed, a striking feature of the *Elements* is how little of it is geometry in our sense: the properties of plane figures are dealt with in Books 1, 3, and 4, and solid geometry is dealt with in Books 11–13.

The remaining books of the *Elements* show how the Greeks used geometry as the basis for all mathematics. Thus Book 2 concerns itself with algebraic propositions (like the expansion $(a + b)^2 = a^2 + b^2 + 2ab$, viewed from a geometric perspective); Books 5 and 6 discuss the theory of proportions and a geometric theory of the real numbers; and Books 7–9 are on number theory. Book 9 includes a proof that the number of primes is infinite, and concludes with a remarkable theorem on perfect numbers (numbers which are the sum of their proper divisors; this definition is believed original with Euclid): If $2^m - 1$ is prime, then $2^{m-1}(2^m - 1)$ is perfect. The longest book in the *Elements* is Book 10, with 115 propositions; in it, Euclid began classifying incommensurable magnitudes.

The Ptolemaic Kingdom reached its greatest cultural heights during the reign of the third Ptolemy, who was such a patron of the arts and sciences that he received the nickname *Euergetes*: benefactor. His reign saw the beginning of a project to translate the Hebrew Bible into Greek. According to tradition, 72 translators (6 from each of the 12 tribes of Israel) were placed in separate cells to translate the Hebrew Bible into Greek; when they completed their work, all 72 translations were identical. Since "duoseptuaginta" (Greek for seventy-two) is hard to pronounce, the version they produced is known as the Septuagint. Textual analysis of the surviving copies (which date to the 4th century A.D.) suggest that, while the translation might have begun during the reign of Ptolemy Euergetes, it was not finished until about a century later.

Ptolemy Euergetes invited ERATOSTHENES (276–196 B.C.) to Alexandria to tutor his son (the future Ptolemy IV). Around 235 B.C., Eratosthenes became head of the library of Alexandria, a post he retained until his death. Very little of Eratosthenes's original work survives; of the surviving work, the most remarkable is his use of simple geometry to measure the circumference of the Earth. The basis of Eratosthenes's calculation was that at Syene, 5000 *stadia* south of Alexandria, the sun was directly overhead at noon on the date of the summer solstice, while in Alexandria, the sun made an angle of about 7° to the vertical. Eratosthenes assumed a spherical Earth and concluded the difference in the angle was caused by the Earth's curvature. Since 7° is approximately $\frac{1}{50}$ of a circle, then the Earth's circumference was about $50 \times 5000 = 250,000$ stadia (see Figure 2.4). A stadia is believed to be about 0.1 miles, so the figure obtained for Earth's circumference by Eratosthenes is within a few percent of the correct value.

Eratosthenes's contemporaries thought highly of his work, and nicknamed him *beta*, after the second letter of the Greek alphabet (which itself comes from the names of the first two letters of the Greek alphabet, α and β). He was given this name because he was always "second best." He was the second best geographer in Alexandria. He was also the second

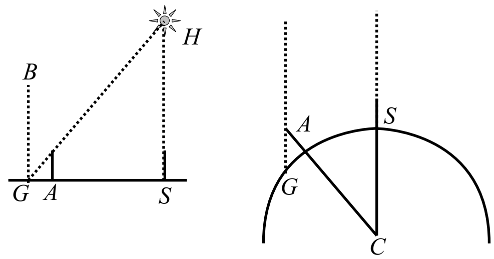

Figure 2.4. Two Explanations for the Angle of the Sun (greatly exaggerated for clarity)

best geometer, historian, astronomer, poet, and literary critic. One can only hope to be as good a "second" as Eratosthenes!

The "alpha" geometer in Alexandria was APOLLONIUS (262–190 B.C.), who came from Perga in Asia Minor. Apollonius himself received the nickname *epsilon* for his work on lunar theory, since the Greek letter ϵ resembles the Moon.

The year before Apollonius was born Eumenes, the governor of Pergamum, revolted against the Seleucids and established an independent kingdom of Pergamum. Apollonius went to Pergamum and stayed a while, though he eventually made his way to Alexandria, probably to study at the museum. His best-known work is his *Conics*, which originally included eight books, though only seven survive. Books I and II were addressed to Eudemus, another geometer in Pergamum, and Books IV through VII of *Conics* were addressed to Attalus, who became king of Pergamum in 241 B.C.

Apollonius investigated the conic sections in such a general and thorough manner that earlier works by Menaechmus and even Euclid were displaced, forgotten, and eventually lost. Euclid defined a cone as the surface of revolution generated by rotating a right triangle about one of its legs; this produces a single-napped, right circular cone. Based on scattered references in other authors, it seems that a conic section was formed by intersecting the cone with a plane perpendicular to the hypotenuse of the triangle. The conic section could be classified, depending on whether the vertex of the cone was acute, right, or obtuse; in modern terms, these sections would be the ellipse, parabola, or hyperbola. Note that since the cone is single-napped, the hyperbola only has one branch.

Instead of revolving a triangle about one of its sides to produce a cone, Apollonius considered instead a circle and a point P not in the same plane. The conic surface would be generated as a line through the point traced along the circle. The point would become the vertex of a double-napped circular cone. Moreover, he considered the intersection of this cone with an arbitrary plane. Besides introducing the conic sections in a new and more general way, Apollonius described and proved many of their properties, including those relating to their tangent lines and propositions concerning the greatest or least distance between a point C and a point on the conic section.

Apollonius completed *Conics* some time after he had moved to Alexandria. By then, Ptolemaic Egypt was in decline. Meanwhile the Seleucids began an expansionist policy under Antiochus the Great and his successors. In 168 B.C. Antiochus Epiphanes ("God manifest," ruled 175–163 B.C.) laid siege to Alexandria. But Egypt had one important ally: Rome. Antiochus was about to take Alexandria when he met an old friend, the Roman ambassador, who told him to leave Egypt or face war with Rome. Antiochus asked for time

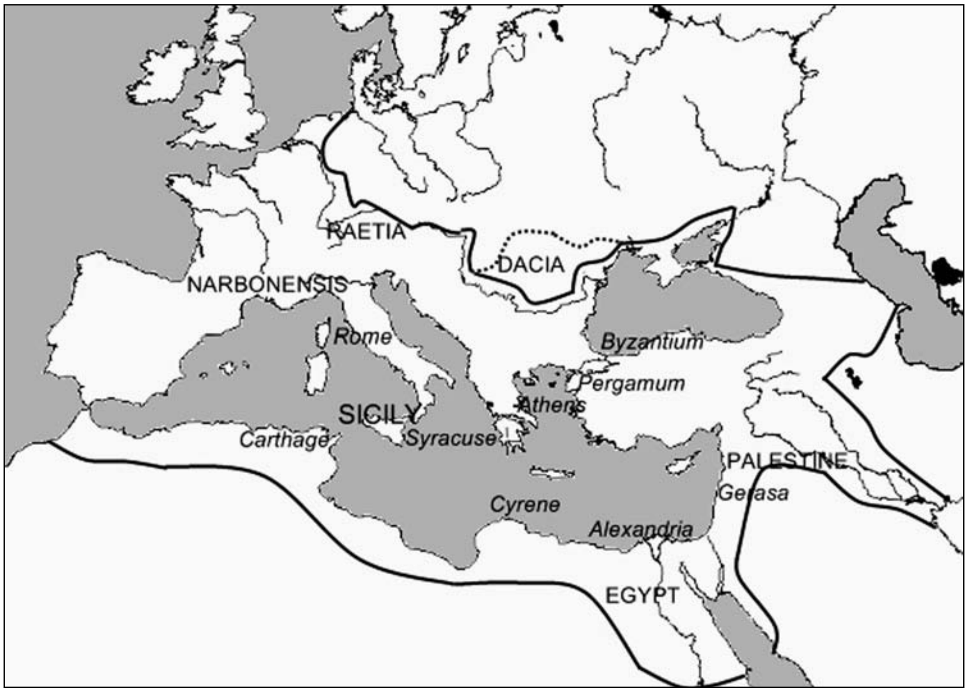

Figure 2.5. The Roman World

to decide, whereupon the Roman ambassador drew a circle in the sand around him and said, "Decide before you leave the circle." Antiochus, with his army around him and a string of victories behind him, was forced to back down before a single, unarmed Roman.

2.2 The Romans

According to tradition, Rome was founded by two brothers, Romulus and Remus, who were raised by a she-wolf. The brothers founded a city on the Tiber River, and shortly thereafter, Romulus killed Remus in a dispute over who got to name the new city. Early Rome was dominated by the Etruscans, a people about whom very little is known. By 509 B.C., the Romans expelled the Etruscan kings and founded the Roman Republic. Over the next few centuries, Rome gradually absorbed its neighbors.

In 282 B.C., Rome invaded Magna Graecia, and the inhabitants of Tarentum called for help from Pyrrhus of Epirus. At Heraclea in 280 B.C., Pyrrhus faced 35,000 Romans with 30,000 of his own soldiers, and a secret weapon, brought from the farthest borders of Alexander's former empire: elephants. The Romans had never before seen elephants, and were forced from the field, losing 7000 men. But Pyrrhus was struck—and concerned—by the bravery of the Roman legionnaires who stood their ground and died, rather than running away. It suggested that the Romans would prove a tough adversary.

At Ausculum in 279 B.C., Pyrrhus defeated the Romans a second time. This time the Romans fought even harder, and Pyrrhus lost so many of his own men that he reputedly said, "Another such victory and I am lost" (the original Pyrrhic victory). When Pyrrhus met Roman soldiers for the third time at Beneventum in 275 B.C., the Romans were ready, and dealt Pyrrhus a severe defeat, elephants and all. Pyrrhus saw nothing to be gained by staying in Italy, so he severed his ties with the Tarentines, who were conquered by Rome in 272 B.C. As he left Sicily, Pyrrhus is said to have remarked, "What a battleground I am leaving for Rome and Carthage."

Carthage was a city on the coast of North Africa (now in Tunisia) founded by the Phoenicians, a seafaring civilization as great as the Greeks and more daring as explorers. According to legend, around 800 B.C. Dido, the daughter of the king of Tyre (in Palestine), had to flee when her brother seized the throne. She made her way to North Africa, where the natives sold her as much land as she could enclose with a bull's hide: this is known as the isoperimetry problem, where one tries to determine the curve that will enclose the greatest area, and is one of the problems examined in the calculus of variations. Dido cut the hide into strips, tied them together, and marked out a semicircle along the coast.

Both Carthage and Rome had designs on Sicily, and an opportunity arose when the Greeks of Syracuse went to war against their old enemies, a group of Italian mercenaries known as the Mamertines ("Sons of Mars"), operating out of Messana. The Syracusans, led by a general named Hieron, drove the Mamertines back to Messana in 270 B.C.; the grateful Syracusans made Hieron their king. In 265 B.C., Hieron made ready to eliminate the Mamertine threat once and for all. Syracuse made an alliance with Carthage, and in response the Mamertines sought help from Rome. What began as a war between Syracuse and Messana turned into a war between Rome and Carthage, the First Punic War (*poeni*, "Phoenician" in Latin).

The war went badly for Syracuse, and in 263 B.C., Hieron made peace with Rome. Thereafter Hieron was one of Rome's staunchest supporters, and to the end of his life, he remained a faithful ally. When the First Punic War ended in 241 B.C., the peace treaty gave control of Sicily to Rome, who gave the eastern half of the island to her faithful ally Hieron and organized the western half of the island as the first province of Rome.

2.2.1 Archimedes

One of Hieron's relatives was ARCHIMEDES (287–212 B.C.), the greatest mathematician of antiquity and possibly the greatest mathematician of all time. As a youth, Archimedes visited Alexandria, probably to study at the museum, and he maintained contact and friendly professional rivalry with some of the geometers and astronomers he met there, notably Eratosthenes and Apollonius. However, he returned to Syracuse where he spent the rest of his life.

Archimedes pursued many different investigations throughout his life. In mathematical physics, he was one of the founding figures of statics and mechanics. In pure mathematics, he was the first to prove the surface and volume formulas for a number of figures, including the sphere and cone. In *The Sand Reckoner*, addressed to Hieron's son Gelon, he calculates the number of particles of sand (dust, really) it would take to fill up the Greek universe; to express the number, he invented octad notation, which has a curious echo more than two thousand years later in the researches of Georg Cantor. In *Quadrature of a Parabola*, he found the area of a segment of a parabola using a geometric sum. In *On Spirals*, he described the construction of a spiral, and proved theorems regarding the tangent to a spiral and the area enclosed by the first turn of a spiral.

The Method of Archimedes is perhaps his most interesting treatise. Archimedes proved many volume relationships, but he wrote *The Method* to provide some insight into how the relationships were discovered in the first place. He gave the details of a means by which problems in mathematics could be investigated using a technique very similar to integral calculus, though he emphasized the results obtained had to be proven "by geometry" (in other words, by a rigorous mathematical method). Essentially, Archimedes considered an n-dimensional figure to consist of an infinite number of $n-1$-dimensional cross-sections; for example, a three-dimensional sphere could be viewed as consisting of an infinite number of two-dimensional disks. These cross sections could then be "weighed" on a balance beam, and in this way a second figure of known properties could be obtained.

Sometime during the Middle Ages, an unknown scribe in Constantinople copied *The Method* onto a piece of vellum. Vellum, made from sheepskins, is very expensive, so the scribe clearly thought very highly of this work. However, vellum has another useful property: it is durable, so that any writing upon it can be washed off and the vellum re-used. In 1229 a copyist named Johannes Myronas erased the text so he could reuse it to write a prayer book; such a recycled piece of vellum is called a *palimpsest*. Generally the erasure is incomplete, and some of the original text can be made out. In 1906, the Danish classicist J. Heiberg heard of a palimpsest in Istanbul that seemed to have originally been a mathematical treatise. Careful examination led to the recovery of the lost work of Archimedes, as well as an accurate date of when the erasure was made and the name of the copyist; in fairness to Myronas, his actions probably saved the text, since as a prayer book, it was kept in relatively good condition.

In addition to his work on quadrature and cubature, Archimedes was the first to consider π in its modern sense as the ratio of the circumference of a circle to its diameter; he also pointed out the correct relationship between the circumference, radius, and area of a circle. Archimedes estimated the value of π by considering the perimeters of circumscribed and inscribed 96-sided polygons and obtained the bounds $3\frac{10}{71} < \pi < 3\frac{1}{7}$. What is most remarkable about his work is his series of rational approximations to irrational numbers. For example, he estimated that $\frac{265}{153} < \sqrt{3} < \frac{1351}{780}$. He presents these approximations without comment, suggesting they (or at least the methods by which they were obtained) were well-known to his correspondents; we know nothing of how they were obtained.

One of the more famous stories about Archimedes concerns his discovery of the principles of hydrostatics. The story of how Archimedes, after trying to determine the volume of Hieron's crown, went to the public baths and had a flash of inspiration that led him to run through the streets crying "Eureka!" ("I have found it!" in Greek) is well known, as is the detail that he neglected to dress on the way out. The last detail ought to be clarified. The Greeks exercised in the nude (the contestants of the Olympic Games, for example, were nude, which is possibly one reason women were not allowed at the games), and the very word *gymnasium* ("to train naked") makes reference to this fact.

Perhaps the most famous story about Archimedes concerns the last few years of his life, when his home city of Syracuse was under siege by the Roman Republic. The siege came about as a result of a new conflict between Rome and Carthage.

Carthage recovered quickly from the First Punic War and built a new empire in Spain; some traces of Carthaginian influence remain to this day, such as the town of Cartagena, named after Carthage. In 218 B.C., after a series of incidents, the Second Punic War began.

The Carthaginian general Hannibal left Spain and crossed the Alps to take the war to Italy, the heartland of the Roman Republic. Hannibal's army, like Pyrrhus's, included elephants, though few survived the crossing of the Alps. Hannibal realized that Rome's greatest advantage was her alliance with the other Italian states. If he could divide Rome from her allies, he could conquer her easily. Thus, he repeatedly stated that his quarrel was with Rome, and with Rome alone. To make this claim convincing, he gave orders that non-Roman soldiers were to be released unharmed, while Romans were shown little mercy.

Hannibal first defeated the Romans at Lake Trasimene, in 217 B.C. The Consul, who led the Roman armies in wartime, was killed. His replacement was Quintus Fabius, who recognized a key weakness in Hannibal's army: it consisted mainly of mercenaries, interested in loot which they could take from a defeated army (which included the captured soldiers themselves, who could be sold as slaves). But if there were no battles, there would be no loot, and the mercenaries would abandon Hannibal for more lucrative ventures. Fabius used the legions to contain Hannibal while patiently waiting for his mercenary army to wither away.

To the Romans, however, the original Fabian policy smacked of cowardice, so Fabius was replaced with Gaius Terentius Varro. In 216 B.C., Varro's 85,000 troops met Hannibal's 50,000 at Cannae. But Hannibal set a brilliant trap for the Roman army, and by day's end, more than 50,000 Roman soldiers were killed, at a cost of 2000 Carthaginians. In all the centuries since, no general has inflicted a greater defeat on his opponent in a single day than Hannibal inflicted on the Romans at Cannae.

Since Hannibal's primary goal was to shatter the Roman confederation, a great victory would be essential: no one would abandon Rome to join the losing side. Cannae was exactly what Hannibal needed, and the Roman confederation began to waver. A key loss would be Sicily, because much of Rome's grain was grown on that island. Hieron stood by Rome, but when the aged king died in 215 B.C. his successor and grandson Hieronymus, abandoned Rome and allied Syracuse with Carthage.

Unfortunately, the Roman confederation proved more resilient than Hannibal imagined, and the core allies of Rome never deserted her. Moreover, at a time when money and equipment from Carthage would have helped Hannibal inflict yet another defeat on Rome, the rulers of Carthage feared what a successful general might do afterwards, so time after time they refused Hannibal's urgent requests for aid. Finally, the Roman people realized that Fabius had been right all along, and put together a new army that would try and contain Hannibal, but not confront him directly. Soon the tide of war turned against Hannibal, and he had to return to North Africa to defend Carthage itself.

Rome turned to deal with her rebellious allies. The pro-Roman faction in Syracuse assassinated Hieronymus and butchered most of the royal family, but when Marcus Claudius Marcellus, the Roman commander in Sicily, stormed the town of Leontini and beheaded 2000 of the troops there for desertion, the Syracusans realized that Rome would never forgive them for joining Carthage. Their choices were victory for Carthage—or annihilation.

In 214 B.C., Marcellus laid siege to Syracuse itself. According to Plutarch, Archimedes made several war machines for Hieron, whose peaceful reign never had need of them. But with the Romans besieging Syracuse, the weapons of Archimedes were put to use. Plutarch recounts tales of ships being sunk by enormous boulders, or lifted out of the water to be dashed on the rocks below, or arrows and darts flung in enormous numbers at Roman soldiers in the field.[10] After a while, the Roman soldiers would flee in terror if they caught sight of a length of rope or a piece of wood coming out of the wall, fearing it might be the precursor to some more horrible attack.

Most of these tales are hard to credit, but it is a matter of record that the siege of Syracuse took two years—somewhat longer than a typical Roman siege. The Romans finally stormed the city in 212 B.C. after the gates were opened by a member of the pro-Roman party. Archimedes was killed during the looting of the city.

According to Plutarch, Marcellus gave orders that Archimedes was to be taken alive. One should not read too much into this order. It was a Roman tradition that a victorious general, such as Marcellus, would be allowed to parade his troops and their trophies of war through the streets of Rome in a procession called a "Triumph." Archimedes, as part of the spoils of war, would probably have been a major display in such a spectacle.

In any case, Plutarch gives three versions of the death of Archimedes. The most likely is that Archimedes was on his way to meeting Marcellus, carrying mathematical and astronomical instruments. Some soldiers, thinking he was a rich citizen fleeing with something valuable, killed and robbed him. Another version is that a soldier came upon Archimedes in the chaos of the sack of the city, and killed him.

The most dramatic and best known version is also the least likely. According to this version, Archimedes was working on a problem when a Roman soldier came up to him and demanded that he accompany him to meet Marcellus. Archimedes refused to leave until he had finished the problem he was working on; the soldier, having no patience for insolence from a civilian, drew his sword and killed him. In the Roman army, insubordination was punishable by a gruesome death, and no Roman soldier would have knowingly disobeyed Marcellus's command. Plutarch reports that the killers of Archimedes were treated as murderers (they were probably executed), and that Marcellus treated Archimedes's surviving relatives with great honor.

The Second Punic War dragged on until 201 B.C., when Rome won a complete victory. The terms of the peace treaty were harsh: Carthage lost Spain and was restricted to North Africa; she was forced to pay an enormous indemnity; and was not to wage war without Rome's permission. Within a generation, she built a new commercial empire, paid off the indemnity, and achieved such success that Rome felt threatened again, this time economically. The anti-Carthaginian party was led by Cato the Elder. After every public speech, regardless of the topic, Cato would add, "And I am also of the opinion that Carthage must be destroyed." Through her allies, Rome provoked the Third Punic War, which resulted in the destruction of Carthage in 146 B.C.

Greece was added to the expanding Roman Empire about the same time. One wonders how the Greeks felt about the conquest. For 350 years, the Greeks fought amongst themselves. The Roman conquest ended centuries of warfare, and for the first time in a dozen generations, Greece was at peace.

2.2.2 The Fall of the Republic

The rest of the Mediterranean would undergo two more centuries of chaos and disruption. Not surprisingly, there was only one mathematician of note in the centuries following Arch-

[10]The infamous "burning mirror" of Archimedes is nowhere mentioned in Plutarch, and does not appear in sources until about four hundred years after the siege of Syracuse.

imedes and Apollonius: HIPPARCHUS of Nicaea (d. ca. 120 B.C.). Hipparchus was unusual for he did *not* work in Alexandria, but spent his life on the Greek island of Rhodes. He was apparently the first person to write a work on chords in a circle, though his work has been lost. The book included a table of chord lengths, the precursor to the modern sine table. By a clever application of geometry and careful observation, Hipparchus correctly determined that the distance to the Moon was thirty times the diameter of the Earth.

In 134 B.C., Hipparchus noticed a star that he had never seen before in the constellation Scorpius (incorrectly called Scorpio by astrologers). Unsure whether this was a new star or one that he simply hadn't noticed, he began to compile the first star catalog, showing the positions of the stars in the sky. He also divided the stars according to their apparent brightness: the twenty brightest stars in the sky were of the "first magnitude"; somewhat dimmer stars were of the "second magnitude", and so on down to the barely visible stars of the "sixth magnitude." The system, with a slight modification to make it more quantitative, is still used today.

While compiling his star chart, Hipparchus also noticed that the position of the sun on the date of the vernal equinox was undergoing a very slow shift that would take about 25,000 years to move it once around the sky. Nearly two thousand years later, the precession of the equinoxes would prove to be one of the key tests of Newton's theory of universal gravitation. One of the consequences of the precession of the equinoxes is that the astrological "signs of the Zodiac," which once corresponded to the constellations in which the sun was located, are now completely unrelated to any physical phenomena.

By the time of Hipparchus the Roman Republic had acquired a vast empire that included most of the Mediterranean, with the exception of the Ptolemaic Kingdom of Egypt. This was a disaster for the middle and lower class. Imported grain was cheaper than domestic grain, which bankrupted the Roman farmer; and slaves acquired as war booty provided labor more cheaply than any Roman citizen could, which wiped out the working class. As a result, the bankrupt farmers were forced to sell their land to the wealthy, who thus acquired huge estates, called *latifundia*, which were worked with slave labor. The working class was reduced to poverty.

In 133 B.C., when Hipparchus was in the midst of compiling his star catalog, Tiberius Sempronius Gracchus was elected to high office. Gracchus promised reform. He was murdered by reactionary nobles in June of that year, establishing a deadly precedent in Roman politics. His brother Gaius tried to continue his program, and met a similar fate (committing suicide before his enemies could murder him). For the next century, Roman fought Roman in a terrible civil war that pitted the aristocratic *optimates* ("best people") against the plebeian *populares* ("rabble rousers").

The leader of the military forces of the *optimates* was Gaius Pompey, a general with an impressive string of victories to his credit. In 67 B.C., pirates made it nearly impossible for Rome to get food shipments from overseas; it took Pompey just six months to wipe out the pirates. Between 66 and 64 B.C., Pompey swept through Asia Minor, and even conquered the troublesome kingdom of Judaea, adding more territory to Rome's vast empire. With proven military leadership and the resources to buy the best arms and armor, the defeat of the *populares* seemed inevitable.

But leading the armed forces of the *populares* was a wealthy aristocrat: Gaius Julius Caesar, who gained notoriety earlier by defeating the Gauls and the Britons. At Pharsalus, Greece, in 48 B.C., Caesar's army, consisting mainly of poorly armed footsoldiers, met Pompey's army, consisting mainly of well-armed cavalry. Caesar demonstrated the true nature of military genius: understanding his opponents. Thus he instructed his footsoldiers to point their spears not at the cavalry horses, which was the standard tactic, but at the faces of the riders. The riders, all aristocrats, had a choice: they could win the battle, but be disfigured; or they could lose. They chose to lose.

Pompey fled to Egypt, hoping to recruit a new army. Egypt was in the midst of a civil war between the twelfth Ptolemy and his older sister. The arrival of Pompey put the regent, Pothinus, in a difficult position. The *optimates* were far from finished, so Pothinus could not afford to antagonize Pompey. But the *populares* had the upper hand, so Pothinus could not afford to antagonize Caesar, either. Pothinus solved his dilemma by having Pompey killed as he disembarked from his ship.

Pothinus showed the body to Caesar and suggested that Caesar return the favor by killing Ptolemy's sister: Cleopatra. The plan backfired, and Caesar threw his forces into the fight on the side of Cleopatra. The alliance of Caesar and Cleopatra has often been romanticized, but Caesar's previous and subsequent behavior showed that, above all else, the pursuit and maintenance of power was his one true love.

It is possible that the Library of Alexandria suffered great destruction during this time. Plutarch claims that Caesar set fire to his fleet as a diversionary tactic; the fire spread and destroyed the Great Library. Tellingly, the geographer Strabo, writing just a few years later, seems to indicate that the works available to him at the Great Library were only a fraction of those available to his predecessors.

Caesar eventually defeated his enemies and emerged the undisputed master of the Roman world. This allowed him to make some necessary changes, including a reform of the calendar. Creating a calendar is a task that lies at the intersection of aesthetics, astronomy, and mathematics. First, one must decide what the calendar is meant to track; a solar calendar tracks the seasons (so, for example, December is always a winter month in the Northern Hemisphere), while a lunar calendar tracks the phases of the Moon (so, for example, the first of the month is always the day of the new Moon), and a luni-solar calendar tracks both (so that the first day of first month of the year is always in the same season and has the same phase of the Moon).

However, the solar year (which determines the length of the seasonal cycle) is 365.2422 days in length, while the synodic month (the time between new Moons) is 29.531 days. Thus the basic mathematical problem of creating a calendar consists of selecting a small number of whole numbers whose mean approximates one of these periods. Most lunar calendars alternate 29 and 30 day months (with a mean of 29.5).

The traditional Roman calendar was a lunar calendar. The year began in March (hence, October was actually the eighth month, as its name suggests) and ended in February. However, twelve lunar months (alternating 29 and 30 days) contain only 354 days, more than 11 days short of the solar year. Without correction, the seasons would drift relative to the calendar: if the beginning of spring occurred in March one year, then three years later it would occur in April. To prevent this, the actual beginning of the year was announced by priests (in fact, the word "calendar" comes from the Greek word meaning "proclamation"). Unfortunately, in Rome these priests were elected officials. This meant they could (and would) delay or accelerate the start of the calendar year, depending on whether or not their party was in power. As a result, by the time of Caesar, the Roman calendar had no relationship to the seasons.

Based on the recommendations of an Egyptian astronomer named Sosigenes, Caesar made several important changes to produce what is now called the Julian calendar. First, to ensure the beginning of the calendar year was actually in the springtime, Caesar ordained 45 B.C. would last 445 days: 45 B.C. came to be known as "The Year of Confusion." Next, Caesar made the lengths of the months alternate, beginning in March with 31 days and April with 30 days (thus abandoning the lunar calendar entirely). The only exception was February, which was an unpopular month: all bills came due in February. Thus, February was shortened to 29 days. Finally, to make the mean calendar year approximate the solar year, every fourth year thereafter would be a leap year, in which February would have 30 days. Note that in the system originally established by Caesar, the months of September, October, November, and December had 31, 30, 31, and 30 days.

Shortly after Caesar's proclamation of the calendar in 45 B.C., he was assassinated on the floor of the Senate. Marc Antony, Caesar's friend, suggested the fifth month be named July, in honor of Caesar's family, the Julians. Caesar's death was followed by another civil war. The major contenders were Caesar's nephew Octavian, and Marc Antony. Marc Antony went to Egypt, to seek help from Cleopatra, while Octavian stayed in Rome and turned the Roman public against Marc Antony. It was not too difficult: Marc Antony had abandoned his wife and was having an affair with Cleopatra.[11]

Marc Antony and Cleopatra's forces suffered a disastrous naval defeat at Actium in 31 B.C. Marc Antony committed suicide. Cleopatra held out hope that she could entrance Octavian as she did Marc Antony and Caesar, but Octavian (something of a stern moralist) refused, offering her only a position in his triumph at Rome—as a captive. She killed herself instead. Octavian swept up the remnants of Ptolemaic Egypt and emerged undisputed master of the Mediterranean. The legions acclaimed Octavian as *Imperator* ("Supreme military commander" in Latin), which became the word Emperor in English.

Octavian was well aware that the last master of the Roman world was murdered on the floor of the Senate. To allay the fears of the Senate, he styled himself "First among equals" and gave himself the title of *Princeps* ("First citizen" in Latin). This became the word prince, and the Roman world became known as the Principate. This pleasant fiction was enough for the Senators, who wanted nothing more than to be treated well. Despite the existence of an elected Senate and Octavian's claim of being merely another citizen, the Republic had ceased to exist, and the Empire had been established.

Octavian took on the name Augustus ("exalted one") and added the name Caesar (to emphasize that he was Caesar's nephew): thus he was the Emperor Augustus Caesar. In time, "Caesar" came to be a title for many ruling monarchs: kaiser, tsar, czar, and (possibly) shah all stem from the word Caesar.

It was during the reign of Augustus that two changes were made to the calendar, which put it almost into its modern form. First, the month following July was named August in honor of Augustus. Then, since August would only have 30 days under the original Julian scheme, a day was taken from February (reducing it to its current 28 days, 29 in leap years). But since the month of September had 31 days, this would leave three 31-day months in

[11] Further circumstantial evidence for the destruction of the Great Library by Caesar's time is that Marc Antony is credited with giving the contents of the Royal Library of Pergamum to Cleopatra as a gift.

a row, so the lengths of the months from September to December were switched to their current amounts. The resulting calendar is now called the Julian calendar.

Augustus was succeeded by his stepson Tiberius, who reigned until 37 A.D. After Tiberius, Roman emperors tended to be weak, incompetent, insane, or all three. Most were murdered or committed suicide before they could be killed by rivals. Nero's suicide in 68 A.D. began the "The Year of Four Emperors," where Galba, Otho, and Vitellius reigned in quick succession, alternately succumbing to assassination or suicide. The "Year of the Four Emperors" ended when Vespasian became Emperor in 69; he ruled for ten years and reformed the Empire's finances, army, and senate. His survival can be traced to a key factor: he was a popular general whose troops had been successfully fighting a revolt in Judea.

Since their conquest by Pompey in 64 B.C., the Jews of Palestine sought independence and eagerly awaited a *messiah* ("deliverer" in Hebrew). One sect believed the delivery would be a religious and spiritual one, while another waited for a great military hero. In either case it was traditional to anoint the deliverer, and in Greek, "the anointed one" is *christos*.

There is little doubt of the existence of a historical Jesus (the Latinized form of the very common Jewish name "Joshua"), though evidence outside the Bible is virtually nonexistent and the best evidence is negative: at no point did contemporary Jewish critics of Christianity ever imply that Jesus was a fictional character. "Joshua the Anointed One" became Latinized to *Jesus Christ*. But because he promised a spiritual and religious salvation, and *not* a military solution, he found little support among the radicals. His supporters came to be known as Christians, and the Romans at first considered Christianity to be another sect of Judaism.

The most outspoken supporter of Christianity was Paul of Tarsus, a Roman citizen who lived around 60. Paul made Christianity more palatable to non-Jews by eliminating circumcision, dietary restrictions, and most of the complex rituals associated with Judaism. By the time of Paul's execution in Rome, Christianity was thriving. Ignatius of Antioch, later canonized as Saint Ignatius, gave the religion a new name: since it was not for Jews or Gentiles, but for everyone, it was *catholic*, which is Greek for "universal."

Meanwhile, the radicals continued to look for a military solution. In 66, the Roman governor of Judea infuriated the Jews by seizing the treasury in the Temple of Solomon to pay back taxes. Protests turned into riots, and riots turned into a full-scale revolt. The radicals, known as Zealots, saw this as an opportunity: why wait for a messiah if, by direct action, they could free themselves? The most fanatical of the Zealots were called Sicariots, (from *sicae*, Greek for "little knife"). The Sicariots believed the best way to deal with Rome was to kill Romans—any Romans, whether they were soldiers, politicians, or innocent bystanders. It has been suggested that Judas *Iscariot*, whom the Bible credits for turning Jesus over to the Roman authorities, is a Hellenization of Judas *Sicariot*. It took three Roman legions under Vespasian to end the revolt.

Vespasian's son Titus captured Jerusalem on September 7, 70. The Temple was destroyed (the "wailing wall" in Jerusalem is all that remains), and the surviving Jews were sold into slavery, beginning the dispersal (*diaspora* in Greek) of the Jews that spread them throughout the Empire. One last band of Zealots held out at the mountain fortress of Masada until 73. The Romans had to build a miles-long ramp to approach the fortress, and before the Romans could break in, the defenders killed themselves.

Clearly military force was not a solution, so there was an increasing turn towards the idea of a spiritual deliverer. Thus it is not surprising that the gospels, which emphasized the spiritual nature of the salvation of the Jews (and, thanks to Paul's work, the non-Jews as well), were written in this period. The gospels (Anglo-Saxon for "good news") were attributed to the disciples of Jesus. However even the earliest to be written down, the Gospel of Mark, was written a generation after the events it claims to describe.

2.2.3 The Return to Order

The Romans built roads, bridges, aqueducts, and buildings that outlived their empire, but theoretical mathematics made little progress during Roman times. This has led to accusations that the only Roman "contribution" to mathematics was killing Archimedes. Even the Roman numeration system is derided as clumsy and inefficient: it is additive, with symbols representing one (I), five (V), ten (X), fifty (L), and so on, so that twenty-three would be XXIII. If a smaller symbol preceded the next larger symbol, this indicated that the latter was to be reduced by the former: thus XL represented ten from fifty, or forty. A more charitable assessment is that Roman numerals were used primarily for record-keeping, where clarity and ease of interpretation are of paramount importance, and subtractive notation was used primarily when the writer ran out of space.

As for Roman contributions to theoretical mathematics, it would be reasonable to claim that theoretical geometry had exhausted the available tools, and progress became nearly impossible. Thus, the mathematically inclined turned to other fields, some of which included purely practical mathematics.

For example, HERON of Alexandria (10–75?) was probably an engineer who taught mathematics: his *Pneumatica* includes descriptions of many clever devices, like a vending machine (5 drachmas for an amount of sanctified water) and a primitive steam engine. Heron's *Metrica* appeared around the same time as the Gospel of Mark, and the subsequent fate of Heron's works mirrors the history of the time. The Gospel of Mark and the Catholic Church grew to become a dominant force in a western civilization that was growing increasingly more interested in philosophy and religion. Meanwhile Greek geometry declined into obscurity, and Heron's *Metrica* disappeared during the Middle Ages, not to be rediscovered until the late nineteenth century.

Metrica concerned itself with the measurement of figures, and in many ways reads like a modern textbook: some results are proven, others are justified non-rigorously, and others are simply attributed. For example, *Metrica* states and proves what is now known as Heron's Theorem: if the sides of a triangle have lengths a, b, and c, and s is half the perimeter, then the area is $A = \sqrt{s(s-a)(s-b)(s-c)}$. To justify the result that the area of a rectangle 5 by 3 units is 15 square units, he notes that the figure can be decomposed into 15 unit squares. The rule:

Rule 2.2. *To find the area of a regular enneagon [9-sided polygon]: Square the side and multiply by* 51*, then divide by* 8*.*

is derived using approximations from a table of chords, and for:

Rule 2.3. *To find the circumference of a circle, multiply the diameter by* 22 *and divide by* 7*. To find the area, take half the product of the circumference and the radius.*

Heron directed the reader to Archimedes.

Heron lived during the relative stability of the reign of Vespasian and his older son Titus. Titus oversaw several important projects, like the building of the *Colosseum* (named after its proximity to an enormous statue of Nero) and the relief effort following the eruption of Vesuvius in 79. Unfortunately he died suddenly in 81 and his younger brother Domitian became emperor. Unlike his father and brother, who coddled the Senators (though allowed them no real power worth speaking of), Domitian treated them like the impotent figures they were. His reign grew increasingly tyrannical, and in 96, he was assassinated in a palace coup whose conspirators included the Empress herself.

The conspirators chose Nerva, a respected Senator, as the new Emperor. He died two years later (of old age), but before his death he chose a conscientious and competent Spanish general, Trajan, to be his successor. This started a trend, and the "Five Good Emperors" (Nerva, Trajan, Hadrian, Antoninus Pius, and Marcus Aurelius) reigned in peaceful succession until 180, each appointing as his successor a man he felt was competent and conscientious.

NICOMACHUS (fl. 100) lived in Gerasa, Palestine (now Jerash, Jordan) around this time, though he probably studied in Alexandria. His *Introduction to Arithmetic* did for number theory what Heron's *Metrica* did for geometry: it presented results and theorems without proof. An important consequence of this is that number theory re-emerged as a subject independent of geometry.

The Pythagoreans developed an elaborate system of number classifications, which Nicomachus gave in detail. The polygonal numbers (from triangular to octagonal) are presented, with an observation equivalent to noting that the second differences of the k-gonal numbers is equal to $k-2$. For example, the triangular numbers are 1, 3, 6, 10, 15, ...; the differences between successive triangular numbers form the sequence 2, 3, 4, 5, ... (the first differences), and the differences between successive terms of these numbers form the sequence 1, 1, 1, 1, ... (the second differences). We can reverse the process to form the n-gonal numbers. For example, the pentagonal numbers would have second differences 3, 3, 3, ..., so their first differences would form the sequence 1, $1+3=4$, $4+3=7$, $7+3=10$, ..., and the pentagonal numbers themselves would form the sequence 1, $1+4=5$, $5+7=12$, $12+10=22$, ... Pyramidal numbers (with triangular and square bases) and other types are presented: the heteromecic numbers, for example, are numbers that are the product of two consecutive whole numbers.

In addition to classifying numbers individually, Nicomachus also classified relationships between numbers. We retain a vestige of this when we refer to one number as a multiple of another, but Nicomachus's classifications go much further. For example, 20 is the triple sesquitertian of 6 (since $20 = 3\cdot 6 + \frac{1}{3}\cdot 6$). As with the polygonal numbers, Nicomachus makes observations on how to form the triple sesquitertians: in this case, the multiples of 10 are the triple sesquitertians of the corresponding multiples of 3 (hence 40, which is 4 times 10, is the triple sesquitertian of 4 times 3).

Another mathematician of the peaceful era of the Five Good Emperors was MENELAUS of Alexandria (fl. 98). His *Sphaerica* is the oldest surviving work to discuss the geometry of triangles on the surface of a sphere. In Book I, Menelaus develops the theory of spherical

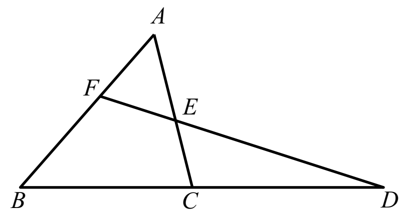

Figure 2.6. Menelaus's Theorem

triangles (defined as regions enclosed by three great circle arcs on the surface of a sphere), and in Book III, he develops some important propositions of spherical trigonometry, including what is known as Menelaus's Theorem. In the plane, let triangles ABC, FDB be as shown (see Figure 2.6). Then $BD \cdot CE \cdot AF = CD \cdot AE \cdot BF$. Menelaus showed that the same would be true if ABC, FDB were spherical triangles (where BD is the chord joining the points B, D).

Even greater contributions to trigonometry were made by CLAUDIUS PTOLEMY (fl. 125–151), who lived during the reign of Hadrian. About all we know for certain about Ptolemy's life is that he made astronomical observations from Alexandria between the years 125 and 151. He would, however, write a book that, like Euclid's *Elements*, incorporated the work of his predecessors in so successful a fashion that tracing the history of trigonometry before Ptolemy is quite difficult. Ptolemy called his work *The Mathematical Collection*, though later commentators called it *The Great Collection*, where "great" is used in the sense of "large." In Greek, this is *Megale Syntaxis*. When Islamic scientists began referring to it, they combined their definite article *al* with the Greek superlative *Megistos*, so the work came to be known as the *Almagest*.

The *Almagest* is mainly known as a work of astronomy, for in it Ptolemy describes a geocentric system that astronomers would use for fifteen hundred years. It highlights a fundamental difference between mathematics and science, for the science in the *Almagest* has been invalidated through the passage of time—but the trigonometry is just as valid and just as usable today as it was during the reign of Hadrian.

The trigonometry of the time was based on the lengths of chords in a circle of standard radius. Ptolemy began the *Almagest* by establishing the necessary theorems, complete with proofs, that allowed him to construct a table of the lengths of the chords in a circle of radius 60. Ptolemy's theorems include those equivalent to the half-angle, angle sum, angle difference, and double angle identities in trigonometry, most of which were derived from what is now known as Ptolemy's Theorem: Let $ABCD$ be a quadrilateral inscribed in a semicircle, with AD the diameter. Then $AC \cdot BD = BC \cdot AD + AB \cdot CD$.

If the radius of the circle is 60, then the chord with a central angle of 60° will have a length of 60 (or 60^P to distinguish it from the angular measure); likewise, we can find the exact length of the chords of 90° and 120°. Thus a combination of the Ptolemaic theorems will give us the lengths of chords of 30°, 45°, and many others. Inconveniently, all of these central angles are multiples of 15°. Fortunately, it is possible to construct a decagon in a circle, and thereby obtain an exact value for the chord with a central angle of 36°. Euclid included a construction, albeit a complex one; Ptolemy gives a very simple and elegant one as follows.

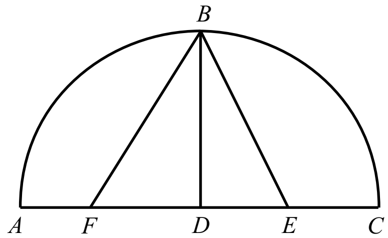

Figure 2.7. Ptolemy's Pentagon

Let AC be a diameter of a circle with center D; let DB be perpendicular to AC. Bisect DC at E and extend BE; let $EF = BE$, and join BF. Then the side of a regular decagon inscribed in the circle will equal FD, and BF will be the side of the regular pentagon inscribed in the same circle (see Figure 2.7).

This will give us the chord with a central angle of 36°, and by applying the various theorems, we can eventually obtain the chords of 3°, $1\frac{1}{2}^{\circ}$, and $\frac{3}{4}^{\circ}$. By using linear interpolation on the last two, Ptolemy found an approximation for the chord of 1°, and from there produced a table of chord values.

Ptolemy also produced another solution to the tuning problem. The difficulty with Pythagorean tuning is that, while most of the the fourths and all of the fifths are perfect, the intervals are inconveniently large. For example, no note between C and F will form a consonance with either note. However, the evidence of the senses points to a peculiarity. Consider an interval spanning three notes (and two whole tones), say C–E. This would be an interval of a third. Since it corresponds to the inelegant ratio 81 : 64, thirds are considered dissonant in Pythagorean music theory. The problem is that if the notes are played melodically (one after the other), they *seem* to be consonant; it is only when they are played simultaneously that their dissonance is obvious.

While 81 : 64 is not a ratio of small whole numbers, it is very nearly equal to the ratio 5 : 4. Thus if we *define* the third to be 5 : 4, we can have another set of consonant intervals. To determine the other ratios, let us begin with C–D, and again let this equal the 9 : 8 ratio that defines an interval. In order for C–E to make a third, D–E must have a ratio of 10 : 9. In order for C–F to make a fourth, E–F must have a ratio of 16 : 15. The F–G interval must be 9 : 8, and so on. Unfortunately, there is a problem; this makes the E–G correspond to 6 : 5 not 5 : 4: it is not a perfect third, but in fact represents a new ratio (again of reasonably small whole numbers). To resolve this problem, we note that in the Pythagorean scheme, the C–E interval includes two whole tones, while E–F includes a tone and a semitone; hence there is no reason to require that they have the same ratio. Unfortunately, this means that we must distinguish between the two types of thirds. A third that includes two tones is a major third, while a third that includes a tone and a semitone is a minor third.

Although a pure Pythagorean would deny the consonance of thirds of any type, they can actually be generated by number mysticism. Consider: the octave can be divided into a fourth and a fifth, either arithmetically (4 : 3 : 2) or harmonically (6 : 4 : 3). The fifth can be divided either arithmetically (6 : 5 : 4) or harmonically (15 : 12 : 10). In both cases, the ratios of 6 : 5 and 5 : 4 are generated, corresponding to the minor and major thirds. Combined with the evidence of the senses that suggests the 6 : 5 and 5 : 4 ratios correspond

Symbol	Diophantus's Term	English	Modern Notation
ς	arithmos	number	x
Δ^Y	dynamos	power	x^2
K^Y	cubos	cube	x^3
$\Delta^Y\Delta$	dynamodynamos	square square	x^4
ΔK^Y	dynamocubos	square cube	x^5
$K^Y K$	cubocubos	cube cube	x^6

Table 1. Diophantus's Notation for Positive Powers

to consonances, the temptation to incorporate thirds is overwhelming. Balancing all of these factors leads to Ptolemaic tuning:

C to D	D to E	E to F	F to G	G to A	A to B	B to c
9 : 8	10 : 9	16 : 15	9 : 8	10 : 9	9 : 8	16 : 15

There are many more consonances here than with Pythagorean tuning, which helped popularize Ptolemaic tuning. Unfortunately, we have sacrificed some of the consonances within the octave. Moreover, if we extend the scale to include two octaves (so, for example, we can consider a fourth whose lower note is A), we discover more dissonances. Consequently Pythagorean tuning also remained popular.

2.2.4 Decline and Fall

The most enigmatic figure in the history of mathematics is DIOPHANTUS of Alexandria, whose *Arithmetic* is the only surviving Greek algebra. The *Arithmetic* consisted of thirteen books, though only ten survive today, six in Greek, and four in Arabic. Most of the problems deal with finding rational solutions to indeterminate equations of the first or second degree, a branch of number theory now called Diophantine analysis.

Diophantus created a type of algebraic notation called syncopated notation, which essentially replaces the words describing an algebraic operation with an abbreviation (see Table 1). He indicated that a number represented a pure number (as opposed to the coefficient) by using the symbol $\overset{\circ}{\mathrm{M}}$. Subtraction was indicated by using a truncated Ψ turned upside down, ⩚.

Diophantus's most famous problem appears in Book II, where he posed the problem of expressing a given square as the sum of two other squares:

Example 2.1. *Let* 16 *be the given square, and suppose it is divided into* x^2 *and* $16 - x^2$, *both assumed squares. Let* $16 - x^2$ *be the square of any number of* x *minus* 4, *the square root of* 16. *Thus:*

$$16 - x^2 = (2x - 4)^2$$
$$16 - x^2 = 4x^2 - 16x + 16$$

$$16x = 5x^2$$
$$x = \frac{16}{5}.$$

Hence the squares are $\frac{256}{25}$ *and* $\frac{144}{25}$.

The dates of Diophantus are very uncertain, though he probably lived during the third century and the dates 200–284 have been suggested for him (though there is some evidence that he predated Heron). Diophantus's life story is given in an epigram:

Problem 2.1. *For one-sixth of his life, Diophantus was a boy. A twelfth part later, he grew a beard, and married after a seventh. Five years later he had a son. The son lived only half the father's life, and Diophantus died four years later.*

It was not unusual for sons to predecease their fathers, but this tragic event was more likely during a time of chaos. This was precisely the state of the Mediterranean world during the third century A.D.

By then a dark change had come over the Empire. Marcus Aurelius, the last of the "Five Good Emperors," chose his son Commodus to be his successor. Not only was Commodus incompetent, he was delusional and believed himself to be the god Hercules. He entered gladiatorial games, killing wild animals in the arena, and even engaged in combats with opponents (the combats were, presumably, rigged). Tired of his excesses, a group of conspirators (including his mistress, Marcia) arranged to have Commodus strangled in 192 by a professional wrestler.

Commodus's successor, Pertinax, was a good man who sought to replenish the treasury by cutting expenditures: in particular, he refused to bribe the Praetorian Guard, the personal bodyguard of the emperor and the most powerful military force in the city of Rome. They killed him, and sold the empire to the highest bidder, a Senator named Marcus Didius Julianus. But Lucius Septimius Severus, a general commanding the Danubian legions, marched on Rome with the intent of declaring himself emperor; the Praetorians, knowing they stood no chance against real soldiers, deposed Julianus as Severus entered Rome. Julianus was executed after holding the throne for only two months, protesting to the end that he had done nothing wrong. If the Praetorians thought their support of Severus would save them, they were wrong: Severus disbanded the guard and established a precedent of military rule. Peace broke out: the peace of the sword.

If Diophantus lived in this era, then he was probably a Roman citizen, thanks to the Edict of Caracalla, proclaimed by Severus's son in 212. The Edict granted Roman citizenship to all freeborn inhabitants of the Roman Empire. By then, Roman citizenship mainly meant you were liable for an inheritance tax; indeed, this has been suggested as Caracalla's motive.

Severus's line came to an end in 235, when a military coup assassinated Alexander Severus (Septimius's grand-nephew), and the legions defending the Rhine acclaimed Maximinus the Emperor. The Senate had their own candidate, who was killed; Maximinus was killed by his own troops in 238. The next forty years saw this cycle repeated many times, with minor variations.

Finally a dour general named Diocletian became emperor in 284. Diocletian made two lasting changes to the empire. To avoid the fate of his predecessors, who reigned an average of two years before being murdered, Diocletian took on the trappings of an Oriental monarch, where the ruler was more god than man. One consequence was that if the emperor was a god, it was treason not to worship him. In particular, this meant the Christians, who refused to recognize any god but their own, were deemed a threat to the safety of the empire. Diocletian began a massive persecution of Christians—the last the Roman world would see.

The other reform of Diocletian was to divide the empire into two halves, an eastern and a western half, to be ruled by *Augusti*. The *Augusti* were equal in theory, though Diocletian made sure that *he* was the "first among equals." The co-emperors would each have co-rulers, called *Caesars*, and together they formed the *tetrarchy* (from the Greek word for "four rulers"). In theory, the Caesars would become emperors, after having gained experience governing. Diocletian's reward for his careful planning: he reigned as absolute master of the Mediterranean for twenty-one years, and retired in 305 to spend the remaining eight years of his life raising vegetables.

To further stabilize the empire, Diocletian mandated that all occupations were to be hereditary: sons had to follow in the professions of their fathers. Certain professions were forbidden entirely; in one edict, Diocletian forbade the practice of mathematics. By this, Diocletian meant the practice of astrology, numerology, and other forms of useless superstition.[12] Indeed, Diocletian's edict goes on to praise the work of *geometers*, who provide useful service to the state. The value placed on geometers can be gauged by their wages, also fixed by edict: an agricultural laborer could earn a maximum of 25 denarii per day, or around 600 denarii per month. For teachers, the pay ranged from 50 denarii per student per month for elementary teachers, 75 for teachers of arithmetic, 100 for teachers of architecture, and 200 for teachers of geometry.

PAPPUS (fl. 300–350) probably lived during the reign of Diocletian. Pappus's main work, his *Mathematical Collection*, was intended for use alongside the great books of the ancient Greek geometers, explaining the more difficult parts to students. There is evidence that the *Mathematical Collections* was meant to contain twelve books, but all that remains is half of Book 2, and Books 3 through 9.

It is possible that Pappus never wrote the last three books. Egypt revolted several times in the 290s. During one revolt, the legions besieged and eventually captured Alexandria but during the siege, a fire was started (no one knows by whom) that consumed half the city, possibly destroying the Great Library (though the annex at the Temple of Sarapis would have been untouched).

Thus the fire may have destroyed the source material Pappus was going to use for the last section of the *Mathematical Collection*. A glimpse of what Pappus had available to him appears in Book 7, where he listed the domain of analysis: those works beyond Euclid's *Elements* that were necessary to learn how to solve geometrical problems. These might be considered the "required reading" for the graduate study of classical mathematics. Table 2 lists the works, their length, and their availability: nine out of the twelve no longer exist. Note that the most advanced work was written by the "beta" mathematician Eratosthenes.

Pappus's intent in writing the *Mathematical Collections* was to revitalize the study of geometry, for he felt geometry had fallen to a very low level. He is extremely critical of

[12] Otho, whose reign was second in the "Year of Four Emperors," also banned astrologers and, according to Suetonius, ordered them to leave Italy by October 1, 69.

Author and Title	Length (Books)	Availability (Books)
Euclid, *Data*	1	1
Apollonius, *Cutting off of a Ratio*	2	2
Apollonius, *Cutting off of an Area*	2	LOST
Apollonius, *Determinate Sections*	2	LOST
Apollonius, *Tangencies*	2	LOST
Euclid, *Porisms*	3	LOST
Apollonius, *Neuses*	2	LOST
Apollonius, *Plane Loci*	2	LOST
Apollonius, *Conics*	8	7
Aristaeus, *Solid Loci*	5	LOST
Euclid, *Loci on Surfaces*	2	LOST
Eratosthenes, *On Means*	2	LOST
Total	33	10

Table 2. The domain of analysis

his fellow geometers, chiding them for doing nothing more than pursuing trivialities and making no new contributions to geometry. At one point, Pappus refers to Pandrosion, who is sometimes credited with being the first female mathematician in history. However, aside from this mention of her in Pappus, where he chides her students for their lack of knowledge, nothing else is known about her. It may be that Pappus invented the name as a way of criticizing a group of Athenian teachers, as Pandrosius was the daughter of Cecrops, the legendary king of Athens.

One of Pappus's original contributions, though he provided no proof, is now called the Pappus-Guldin theorem: the volume of a solid of revolution can be found by multiplying the area of the figure by the distance traveled by its center of mass. In the fifth book of the *Collection* Pappus solved the isoperimetry problem by showing that of all figures of a given perimeter, the circle has the greatest area.

If Pappus really lived in Alexandria during the reign of Diocletian, he would have witnessed the beginning of a crucial battle within the Church. An Alexandrian priest, Arius claimed that Jesus Christ was not God, but was instead created by God, a doctrine now called Arianism. Arius's main opponent was the Bishop of Alexandria, Athanasius, and as the fourth century opened, the conflict between Arians and Athanasians loomed.

The conflict between Arian and Athanasian might have been inconsequential had Diocletian's tetrarchy kept the peace. Unfortunately, the tetrarchy only lasted one generation before a new round of civil wars began. It ended when Constantine became sole emperor in 313. Before the Battle of Milvian Bridge (312), Constantine was said to have seen a flaming cross, with the words, "With This Sign, Conquer." Constantine had his troops paint the symbol on their shields, and won the day. Though it is claimed that Constantine also converted to Christianity after the battle, he was not baptized (if he ever was) until many years later; however, he did soften the official stand on Christianity. In 313, he proclaimed the Edict of Milan, which officially ended the persecution of Christianity. Constantine also sponsored the Council of Nicaea in 325 to mediate some of the doctrinal disputes within the increasingly influential Catholic church. The Council of Nicaea declared Arianism to be heretical, but in spite of this, the Emperors of Rome were alternately Arians (who censured and exiled Athanasius) and Orthodox (who allowed Athanasius to return).

Constantine moved the capital of the Empire to Byzantium. Surrounded on three sides by water, it was virtually impossible to capture, and was one of the few cities that withstood Philip of Macedon. After much new construction to make it a proper capital city, it was dedicated in 330 and given a new name: Constantinople ("Constantine's City", in Greek).

In 379, the Orthodox Theodosius became Emperor, and the Christian Church began to play an increasingly pivotal role in the governance of the Empire. Key among those who extended the power of the Church over the state was Ambrose, Bishop of Milan. In 384, Ambrose persuaded the Emperor Theodosius to reject an appeal for tolerance by the pagans. Four years later, the Emperor punished a Bishop who had burned down a Jewish synagogue, and Ambrose rebuked him for it. Intolerance for non-Christians was rapidly gaining a foothold in the Empire, and pagan learning became increasingly suspect.

For example Jerome, who had traveled about the Mediterranean studying Greek and Latin classics, had a vision that he was brought before God and accused of being a follower of Cicero, not Christ. As a result, he turned his enormous intellectual talents to religious pursuits. At the time, the version of the Bible in use was the Greek Septuagint. But few Christians could read Greek, so from 391 to 407, Jerome prepared a Latin version of the Bible. Since it was meant for the people (*vulgis* in Latin), it became known as the Vulgate and is the basis for the Catholic Bible.

Not all Christians were implacably opposed to pagan learning. Ambrose studied the Greek classics extensively, incorporating elements of Greek philosophy to make Christianity more palatable to philosophically minded pagans. This helped Ambrose convert Augustine, a Neo-Platonist, around 387; Augustine went on to become Bishop of Hippo in North Africa in 396, and is considered one of the major Christian philosophers. In his longest work, the *Ennarations* (sermons based on the *Psalms*), Augustine lists those who consult mathematicians among God's enemies. Since he lists mathematicians alongside sorcerers and pagan oracles, it is clear that he is referring to mathematicians in the Diocletian sense.[13]

The case of Synesius of Cyrene (in North Africa) is particularly interesting. Synesius was married and had three children; this did not prevent him from becoming the Bishop of Ptolemais. Earlier, he had studied at a Neo-Platonic Academy in Alexandria, and maintained contact with his old teacher: HYPATIA (ca. 370–March 415), the daughter of THEON of Alexandria (ca. 335–405).

Hypatia, as the first known female mathematician in history (with the possible exception of Pandrosion), has been so mythologized that it is difficult to separate fact from fantasy. The definite facts are that she helped Theon with his commentary on Ptolemy, and became the head of the Neo-Platonist school in Alexandria around 400, teaching mathematics and philosophy. She also wrote a commentary on Diophantus's work, which is believed to be the source of the six surviving Greek books of his *Arithmetic*.

In 391, the Bishop of Alexandria, Theophilus, with the support of the Emperor Theodosius, incited Christian mobs to destroy the remaining pagan temples. These included

[13] Interestingly enough, he does not attack mathematicians as such: only those who use them, because they are placing worldly concerns over spiritual ones.

the Temple of Sarapis, where the last surviving manuscripts of the once great Library of Alexandria were kept. Theophilus's nephew and successor Cyril continued his uncle's destructive work. Cyril encouraged his followers to massacre the Jews, which destroyed a population that had been living peacefully and prosperously in Egypt since the time of the first Ptolemy.

Such flagrant violations of the peace had to be addressed by Orestes, the governor of Egypt. Though Orestes was Christian himself, he could not, as governor, allow one segment of the population to terrorize the rest. However, Orestes dared not move directly against Cyril, so he appealed to Pulcheria, the regent for the Emperor Theodosius II. She declined to move directly against the Christians, so this appeal was useless. Indeed, even the attempt to move against Cyril caused Orestes to be assaulted by a group of fanatics. The chief attacker, Ammonius, was executed for the assault, but Cyril had his body exhumed and treated as if Ammonius were a martyr.

It was suggested to Cyril that the death of Hypatia, Orestes's friend and supporter, would lead to a reconciliation between Cyril and Orestes. Thus in March 415, she was set upon by a fanatical mob of Christian *parabalani* ("lay brethren") led by a lector named Peter who dragged her into the street, stripped her naked, then flayed the flesh from her body with shells (or tiles) before burning her body.

Clearly the intellectual environment in fifth century Alexandria had turned against pagan philosophy. Despite this, PROCLUS (410–485) came to Alexandria to study philosophy. Not surprisingly, he found the instruction poor, and moved to Athens around 430 to continue his study at Plato's Academy with Syrianus of Alexandria. When Syranius died in 437, Proclus became the head of the Academy, earning him the nickname *Diadochus* ("successor"). He wrote a number of works on philosophy and composed music, but his greatest contribution was his commentary on the first book of Euclid's *Elements*. Proclus gives us invaluable information on the development of Greek mathematics, partly because he had access to works now lost, including a history of geometry written by EUDEMUS (fl. 300 B.C.) and a comprehensive encyclopedia on mathematics written by GEMINUS (fl. 60 B.C.).

Proclus never married, though he had been engaged to Aedesia of Alexandria. However, an oracle foretold disaster if the wedding occurred, so she married Hermeias, another student of Syrianus. Hermeias taught Platonism at a Neoplatonic academy in Alexandria, indicating that pagan learning had not been completely suppressed, but died around 445. After his death, Aedesia moved to Athens with her sons, Ammonius and Heliodorus; subsequently, she sent them to study with Proclus.

Aedesia and her sons returned to Alexandria around 475, where Ammonius took up his father's former post after reaching some sort of arrangement with the Bishop of Alexandria, Peter III Mongus. The nature of the deal with the Bishop is unknown: suggestions range from modifying Neoplatonic doctrine so it supported Christianity to betraying the hiding places of other pagans. Most likely, Ammonius agreed to teach only non-offensive subjects.

Ammonius's continued teaching proved crucial to the history of mathematics, for one of his students was EUTOCIUS (ca. 480–ca. 540). Eutocius wrote commentaries on Archimedes's *On the Sphere and Cylinder*, *Measurement of a Circle*, *On Plane Equilibria*, as well on the first four books of Apollonius's *Conics*. Without these commentaries, our knowledge of the early history of Greek geometry would be much poorer; and in the case of Apollonius's *Conics*, we might not even have the Greek text.

Rome was even more backwards than Alexandria or Athens. By 500, the Italian peninsula was part of the Kingdom of the Ostrogoths under Theodoric the Great. The Kingdom of the Ostrogoths preserved some of the forms of Roman government, such as the consulship, though by then the position was largely symbolic. One consul was ANICIUS MANLIUS SEVERINUS BOETHIUS (ca. 480–524). After a disagreement with Theodoric, Boethius was imprisoned and then executed. While in prison, Boethius wrote *The Consolation of Philosophy*. Though Boethius himself was not a Christian, this work became one of the cornerstones of medieval Christian philosophy.

Boethius also translated many Greek works into Latin, including mathematical texts. However, his translations show the depths to which classical learning had fallen. His version of the *Elements* has been lost, but references to it suggest that it included only the simplest results, and omitted all proofs. Boethius also wrote an account of Nicomachus's *Arithmetic*, giving an account of Pythagorean number theory. These were meant to be part of a series of handbooks on the four mathematical sciences: arithmetic, geometry, music, and astronomy. These four subjects formed the *quadrivium* ("four roads" in Latin), and are roughly analogous to the course of study for a modern baccalaureate degree. Study of the quadrivium assumed a study of the *trivium* ("three roads" in Latin): grammar (Latin), rhetoric, and logic. Because the trivium was perceived to be easier than the quadrivium, the word "trivial" came to be applied to any simple study.[14]

Five years after Boethius's death, the last light of classical learning was put out when the Emperor Justinian closed the schools of pagan learning in Athens and elsewhere in 529. He also passed an edict that banned pagans from teaching, public office, and military service. Learning in the West declined precipitously, which entered a "Dark Age." Fortunately, learning and scholarship existed and thrived elsewhere.

For Further Reading

For the history of Greece and Rome, see [1, 11, 34, 60, 63]. For mathematics and science, see [48, 46, 49, 47, 84].

[14] Augustine began a similar project before his conversion to Christianity.