Law III

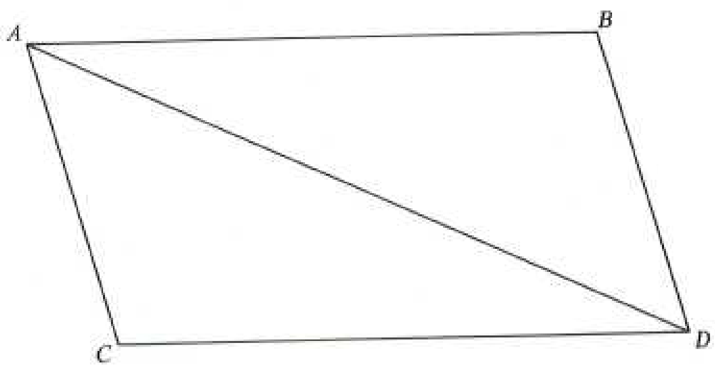
To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

Whatever draws or presses another is as much drawn or pressed by that other. If you press a stone with your finger, the finger is also pressed by the stone. [. . .] If a body impinge upon another, and by its force change the motion of the other, that body also (because of the equality of the mutual pressure) will undergo an equal change, in its own motion, towards the contrary part. The changes made by these actions are equal, not in the velocities but in the motions of bodies; that is to say, if the bodies are not hindered by any other impediments. For, because the motions are equally changed, the changes of the velocities made towards contrary parts are inversely proportional to the bodies. This law takes place also in attractions, as will be proved in the next Scholium.

Corollary I

A body, acted on by two forces simultaneously, will describe the diagonal of a parallelogram in the same time as it would describe the sides by those forces separately.

If a body in a given time, by the force M impressed apart in the place A, should with an uniform motion be carried from A to B, and by the force N impressed apart in the same place, should be carried from A to C, let the parallelogram $ABCD$ be completed, and, by both forces acting together, it will in the same time be carried in the diagonal



from A to D. For since the force N acts in the direction of the line AC, parallel to BD, this force (by the second Law) will not at all alter the velocity generated by the other force M, by which the body is carried towards the line BD. The body therefore will arrive at the line BD in the same time, whether the force N be impressed or not; and therefore at the end of that time it will be found somewhere in the line BD. By the same argument, at the end of the same time it will be found somewhere in the line CD. Therefore it will be found in the point D, where both lines meet. But it will move in a right line from A to D, by Law I.

12.B3 The method of first and last ratios

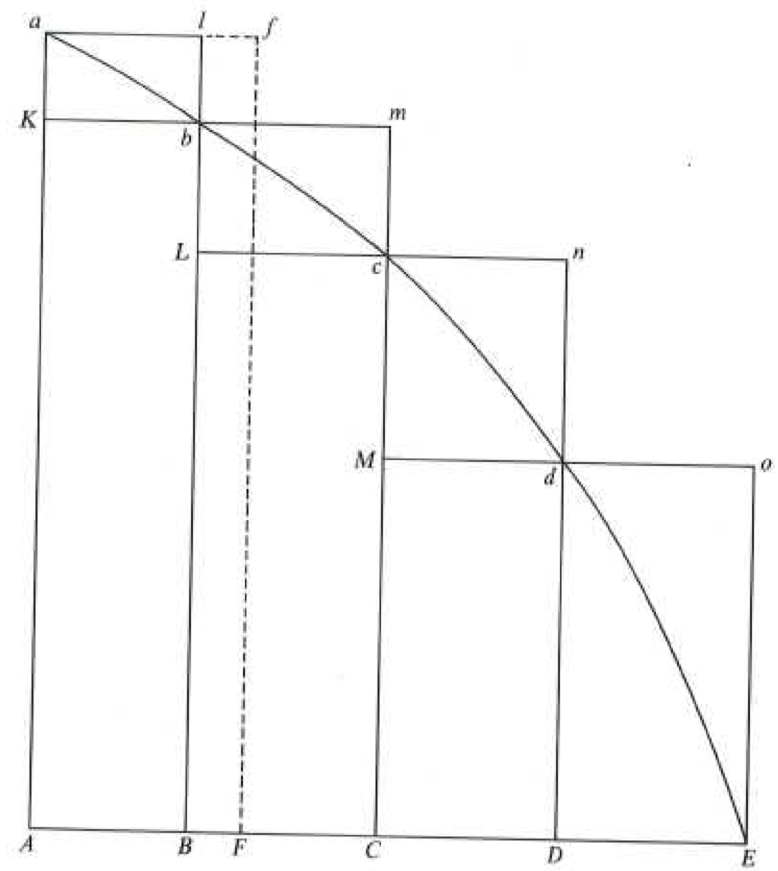
Lemma I

Quantities, and the ratios of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer to each other than by any given difference, become ultimately equal.

If you deny it, suppose them to be ultimately unequal, and let D be their ultimate difference. Therefore they cannot approach nearer to equality than by that given difference D; which is contrary to the supposition.

Lemma II

If in any figure AacE, terminated by the right lines Aa, AE, and the curve acE, there be inscribed any number of parallelograms Ab, Bc, Cd, &c., comprehended under equal bases AB, BC, CD, &c., and the sides, Bb, Cc, Dd, &c., parallel to one side Aa of the figure; and the parallelograms aKbl, bLcm, cMdn, &c., are completed: then if the breadth of those parallelograms be supposed to be diminished, and their number to be augmented in infinitum, *I say, that the ultimate ratios which the inscribed figure*



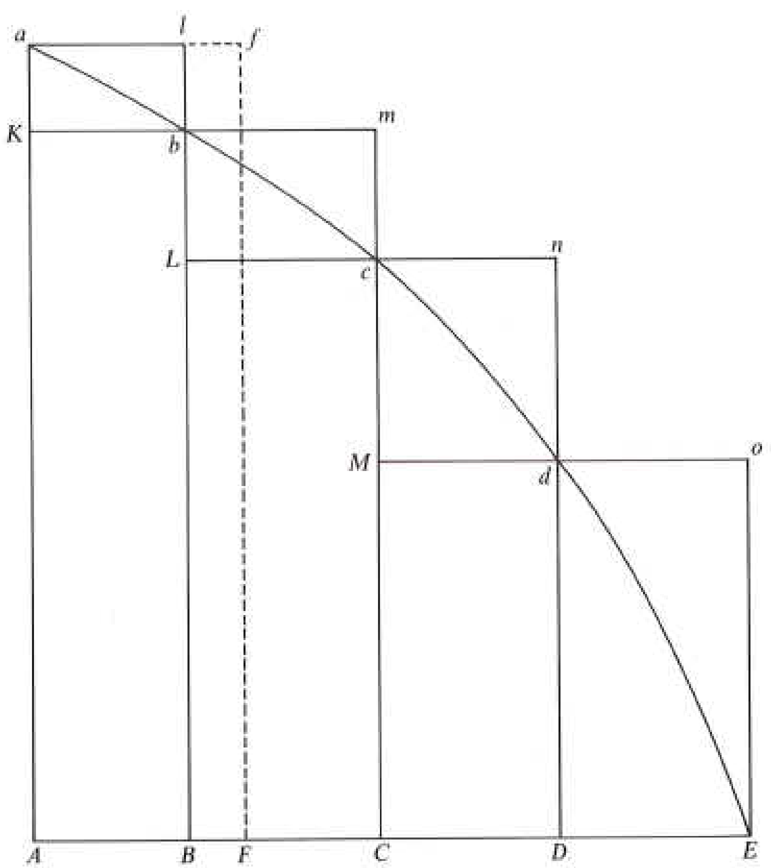
AKbLcMdD, the circumscribed figure AalbmcndoE, and curvilinear figure AabcdE, will have to one another, are ratios of equality.

For the difference of the inscribed and circumscribed figures is the sum of the parallelograms *Kl*, *Lm*, *Mn*, *Do*, that is (from the equality of all their bases), the rectangle under one of their bases *Kb* and the sum of their altitudes *Aa*, that is, the rectangle *ABla*. But this rectangle, because its breadth *AB* is supposed diminished *in infinitum*, becomes less than any given space. And therefore (by Lem. I) the figures inscribed and circumscribed become ultimately equal one to the other; and much more will the intermediate curvilinear figure be ultimately equal to either. Q.E.D.

Lemma III

The same ultimate ratios are also ratios of equality, when the breadths AB, BC, DC, &c., of the parallelograms are unequal, and are all diminished in infinitum.

For suppose *AF* equal to the greatest breadth, and complete the parallelogram *FAaf*. This parallelogram will be greater than the difference of the inscribed and circumscribed figures; but, because its breadth *AF* is diminished *in infinitum*, it will become less than any given rectangle. Q.E.D.



Corollary I Hence the ultimate sum of those evanescent parallelograms will in all parts coincide with the curvilinear figure.

Corollary II Much more will the rectilinear figure comprehended under the chords of the evanescent arcs *ab*, *bc*, *cd*, &c., ultimately coincide with the curvilinear figure.

Corollary III And also the circumscribed rectilinear figure comprehended under the tangents of the same arcs.

Corollary IV And therefore these ultimate figures (as to their perimeters *acE*) are not rectilinear, but curvilinear limits of rectilinear figures.

12.B4 The nature of first and last ratios

Those things which have been demonstrated of curved lines, and the surfaces which they comprehend, may be easily applied to the curved surfaces and contents of solids. These Lemmas are premised to avoid the tediousness of deducing involved demonstrations *ad absurdum*, according to the method of the ancient geometers. For demonstrations are shorter by the method of indivisibles; but because the hypothesis of indivisibles seems somewhat harsh, and therefore that method is reckoned less geometrical, I chose rather to reduce the demonstrations of the following Propositions to the first and last sums and ratios of nascent and evanescent quantities, that is, to the limits of those sums and ratios, and so to premise, as short as I could, the demonstrations of those limits. For hereby the same thing is performed as by the method of indivisibles; and now those principles being demonstrated, we may use them with greater safety. Therefore if hereafter I should happen to consider quantities as made up of particles, or should use little curved lines for right ones, I would not be understood to mean indivisibles, but evanescent divisible quantities; not the sums and ratios of determinate parts, but always the limits of sums and ratios; and that the force of such demonstrations always depends on the method laid down in the foregoing Lemmas.

Perhaps it may be objected, that there is no ultimate proportion of evanescent quantities; because the proportion, before the quantities have vanished, is not the ultimate, and when they are vanished, is none. But by the same argument it may be alleged that a body arriving at a certain place, and there stopping, has no ultimate velocity; because the velocity, before the body comes to the place, is not its ultimate velocity; when it has arrived, there is none. But the answer is easy; for by the ultimate velocity is meant that with which the body is moved, neither before it arrives at its last place and the motion ceases, nor after, but at the very instant it arrives; that is, that velocity with which the body arrives at its last place, and with which the motion ceases. And in like manner, by the ultimate ratio of evanescent quantities is to be understood the ratio of the quantities not before they vanish, nor afterwards, but with which they vanish. In like manner the first ratio of nascent quantities is that with which they begin to be. And the first or last sum is that with which they begin and cease to be (or to be augmented or diminished). There is a limit which the velocity at the end of the motion may attain, but not exceed. This is the ultimate velocity. And there is the like limit in all quantities and proportions that begin and cease to be. And since such limits are certain and definite, to determine the same is a problem strictly geometrical. But whatever is geometrical we may use in determining and demonstrating any other thing that is also geometrical.

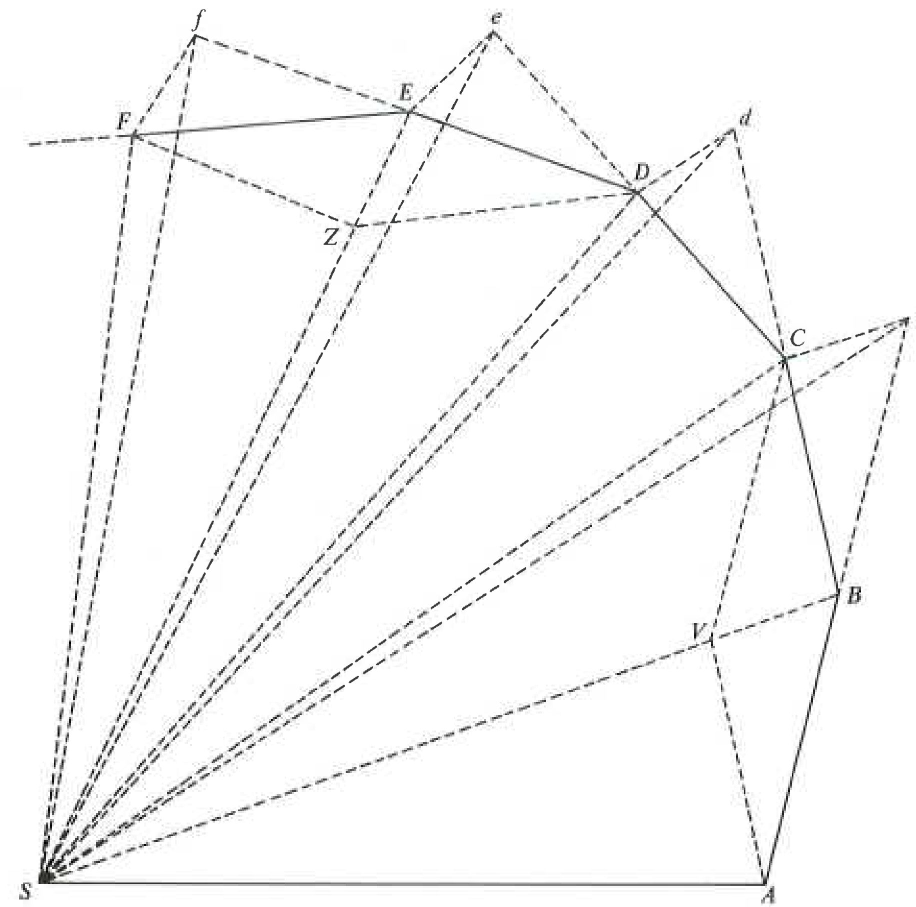
It may also be objected, that if the ultimate ratios of evanescent quantities are given, their ultimate magnitudes will be also given: and so all quantities will consist of indivisibles, which is contrary to what Euclid has demonstrated concerning incommensurables, in the tenth Book of his *Elements*. But this objection is founded on a false supposition. For those ultimate ratios with which quantities vanish are not truly the ratios of ultimate quantities, but limits towards which the ratios of quantities decreasing without limit do always converge; and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminished *in infinitum*. This thing will appear more evident in quantities infinitely great. If two quantities, whose difference is given, be augmented *in infinitum*, the ultimate ratio of these quantities will be given, namely, the ratio of equality; but it does not from thence follow, that the ultimate or greatest quantities themselves, whose ratio that is, will be given. Therefore if in what follows, for the sake of being more easily understood, I should happen to mention quantities as least, or evanescent, or ultimate, you are not to suppose that quantities of any determinate magnitude are meant, but such as are conceived to be always diminished without end.

12.B5 The determination of centripetal forces

Book I, Proposition I, Theorem I

The areas which revolving bodies describe by radii drawn to an immovable centre of force do lie in the same immovable planes, and are proportional to the times in which they are described.

For suppose the time to be divided into equal parts, and in the first part of that time let the body by its innate force describe the right line *AB*. In the second part of that time, the same would (by Law I), if not hindered, proceed directly to *c*, along the line *Bc* equal to *AB*; so that by the radii *AS*, *BS*, *cS*, drawn to the centre, the equal areas *ASB*, *BSc*, would be described. But when the body is arrived at *B*, suppose that a centripetal force acts at once with a great impulse, and, turning aside the body from the right line *Bc*, compels it afterwards to continue its motion along the right line *BC*. Draw *cC* parallel to *BS*, meeting *BC* in *C*; and at the end of the second part of the time, the body (by Cor. I of the Laws) will be found in *C*, in the same plane with the triangle *ASB*. Join *SC*, and, because *SB* and *Cc* are parallel, the triangle *SBC* will be equal to the triangle *SBc*, and therefore also to the triangle *SAB*. By the like argument, if the centripetal force acts successively in *C*, *D*, *E*, &c., and makes the body, in each single particle of time, to describe the right lines *CD*, *DE*, *EF*, &c., they will all lie in the same plane; and the triangle *SCD* will be equal to the triangle *SBC*, and *SDE* to *SCD*, and *SEF* to *SDE*. And therefore, in equal times, equal areas are described in one immovable plane: and, by composition, any sums *SADS*, *SAFS*, of those areas, are to each other as the times in which they are described. Now let the number of those triangles be augmented, and their breadth diminished *in infinitum*; and (by Cor. IV, Lem. III) their



ultimate perimeter *ADF* will be a curved line: and therefore the centripetal force, by which the body is continually drawn back from the tangent of this curve, will act continually; and any described areas *SADS*, *SAFS*, which are always proportional to the times of description, will, in this case also, be proportional to those times. Q.E.D.

Proposition II, Theorem II

Every body that moves in any curved line described in a plane, and by a radius drawn to a point either immovable, or moving forwards with an uniform rectilinear motion, describes about that point areas proportional to the times, is urged by a centripetal force directed to that point.

Case 1 For every body that moves in a curved line is (by Law I) turned aside from its rectilinear course by the action of some force that impels it. And that force by which the body is turned off from its rectilinear course, and is made to describe, in equal times, the equal least triangles *SAB*, *SBC*, *SCD*, &c., about the immovable point *S* (by Euclid's *Elements*, I.40 and Law II), acts in the place *B*, according to the direction of a line parallel to *cC*, that is, in the direction of the line *BS*; and in the place *C*, according