6.C The Foundations of Geometry

6.C1 Al-Haytham on the parallel postulate

Al-Haytham proposed to prove the postulate (Postulate 5 of Euclid's *Elements*, see 3.B1) using only the first twenty-eight propositions of Euclid's *Elements*, which, as he observed, are independent of it.

Let us start with a premise for that, and that is: 'When two straight lines are produced from the two extremities of a finite straight line, containing two right angles with the first line, then every perpendicular dropped from one of these two lines on the other is equal to the first line, which contained two right angles with these two lines.' Thus, every perpendicular dropped from one of the afore-mentioned lines on the other contains a right angle with the line from which it was dropped. An example of this is as follows: there is extended from the two extremities *A*,*B* of line *AB* two lines *AG*, *BD*, and the two angles *GAB*, *DBA* are each right. Then point *G* is assumed on line *AG* and from it perpendicular *GD* is dropped on line *BD*. I say, then, that line *GD* is equal to line *AB*. The proof of that is that nothing else is possible.

If it were possible, then let it not be equal. If *GD* is not equal to *AB*, it is either greater than it or less. Let it first be greater than it. Line *GA* is produced in a straight line in the direction of *A*, and let it be *AE*. *BD* is also extended rectilinearly in the direction of *B*, and let it be *BT*. We cut off *EA* equal to *AG*. From point *E* a perpendicular is dropped on line *BT*, and let it be *ET*. Let us connect the two lines *GB*, *BE*. Since line *GA* is equal to line *EA* and line *AB* is common, the two lines *GA*, *AB* are equal to the two lines *EA*, *AB* and angles *GAB*, *EAB* are equal because they are right. Therefore, base *GB* is equal to base *EB*, and triangle *GAB* is equal to triangle *EAB*, and the rest of the angles are equal to the rest of the angles. Therefore angle *GBA* is equal to angle *EBA*; the whole angle *ABD* is equal to the whole angle *ABT*. It remains that angle *GBD* is equal to angle *EBT*, and angle *GDB* is equal to angle *ETB*, since they are right. Therefore, triangle *GDB* is equal to triangle *ETB* since two angles of one of them are equal to two angles of the other, and the two sides *GB*, *BE* are equal. Therefore, line *GD* is equal to line *ET*. But *GD* had been greater than *AB*, so *ET* is greater than *AB*.

Let us imagine line *ET* moving along line *TB*, while, during its motion, it is perpendicular to it, so that angle *ETB*, throughout the motion of *ET*, is always right. When point *T*, by the movement of line *ET* ends up at point *B*, line *ET* will coincide with line *BA*, since the two angles *ETB*, *ABD* are equal (because each of them is a right angle). When line *ET* coincides with line *BA*, point *E* will be outside line *AB* and higher than point *A*, since line *ET* had been shown as being greater than line *AB*. Therefore, let line *ET*, while it coincides with line *BA* be line *BH*. Line *BH* can be imagined also after this position moving in the direction of *GD*, and it is in the equivalent of its first position. Then, when point *B*, by the motion of line *BH* ends up at point *D*, line *BH* coincides with line *DG* since the two angles *HBT*, *GDB* are equal because they are right. When line *BH* coincides with line *GD*, point *H* coincides with point *G*, since line *HB* is line *ET* and line *ET* is equal to line *GD*. When line *BH* (which is line *ET*) arrives at line

GD and coincides with it, line *ET* will have moved over line *TD*, and point *T* will have ended up at point *D*. Point *E* will have ended up at point *G*. It was shown above in defining parallel lines that the higher end of every line moving in this way traces a straight line; therefore point *E* traces a straight line during the movement of line *ET* over line *TB*. Let the line point *E* traces be line *EHG*; thus, line *EHG* is a straight line, but line *EAG* is a straight line by assumption. Point *H* has been shown to be higher than point *A*, so line *EHG* is other than line *EAG*. But the two points *E*,*G* are common to these two lines and are straight, therefore two straight lines would contain a space; but this is impossible. The impossibility necessarily follows from our assumption that line *GD* is greater than line *AB*. Therefore, line *GD* cannot be greater than line *AB*.

[Similarly he then showed that it could not be less.]

6.C2 Omar Khayyam's critique of al-Haytham

A part of wisdom, the easiest one, is called mathematics. Few of the matters are quite obvious, but sometimes in geometry a simple matter hides even from a sound and keen mind and an excellent intuition.

This part of wisdom, mathematics, is based on a book of wisdom called logic. It discusses things based on common sense such as that the whole is larger than a part of it. But for axioms there is no proof from common sense.

For a long time I had a strong desire in studying and research in sciences to distinguish some from others, particularly, the book [Euclid's] *Elements of Geometry* which is the origin of all mathematics, and discusses point, line, surface, angle, etc. There are many postulates which should be accepted without proof, such as through two given points there passes one and only one straight line. But there are doubtful matters, among them, the greatest one which has never been proved, i.e., 'Two straight lines intersect if they meet a given line in two distinct points such that the sum of the angles on the one side of the given line between the two points is less than two right angles', has been taken to be true.

I have seen many books which have objected to this idea, among the earlier ones Heron and Autolycus, and the later ones al-Khazen, al-Sheni, al-Neyrizi, etc. None has given a proof. Then I have seen the book of Ibn Haytham, God bless his soul, called the solution of doubt. This postulate among other things was accepted without proof. There are many other things which are foreign to this field such as: If a straight line segment moves so that it remains perpendicular to a given line, and one end of it remains on the given line, then the other end of it draws a parallel.

There are many things wrong here. How could a line move remaining normal to a given line? How could a proof be based on this idea? How could geometry and motion be connected? Motion is only allowed for a single element. A line is generated by a motion of a point and a surface is generated by a motion of a line. Euclid says that a sphere is generated by rotation of a half circle. This solid is bounded by a surface whose points are equidistant to a fixed point inside of it. Also, Euclid uses a straight line segment with a fixed end. He rotates this line segment around the fixed end of it, in a flat surface, to get a circle. But none of these is comparable with Ibn Haytham's idea.