Chapter IV

Arabic Primacy with Hindu, Chinese, and Maya Contributions

(Muhammad ibn Mūsā) al-Khwārizmī (fl. 800-847)

Although his ancestors came from Khorezm (which corresponds to the modern city of Khiva and its surrounding district in the northwestern Uzbek Soviet Socialist Republic south of the Aral Sea in central Asia), al-Khwārizmī probably grew up near Baghdad and was an orthodox Muslim. Under Caliph al-Ma'mūn (reigned 813-833), a patron of humanistic and scientific learning, he became a member of the "House of Wisdom" (Bayt al-Hikmah), a kind of science academy in Baghdad. He thus lived and worked at the intellectual and cultural center of Islamic civilization, when it was beginning to assimilate Greek and Hindu science. For al-Ma'mūn he prepared an astronomical treatise, which he based on the Sanskrit astronomical research of Brahmagupta, and dedicated his book *Algebra* to that caliph. He was probably one of the astronomers called to cast a horoscope for Caliph al-Wāthiq (d. 847) when the latter was dying.

Al-Khwārizmī conducted research on topics other than astronomy and algebra. His text *Geography*, which he seems to have based partly on Ptolemy's *Geography*, contained more accurate maps of Islamic areas than Ptolemy's. (Islamic geography was important since faithful Muslims faced Mecca at prayer.) He also authored treatises on Hindu numerals and on the Jewish calendar and three lost books entitled *Book on the Construction of the Astrolabe*, *On the Sundial*, and *Chronicle* (a historical text on the events of the early Islamic period). His scientific and mathematical writings—all mediocre at best—were uncommonly influential because they transmitted new knowledge to his successors in the high period of Islamic culture.

The *Algebra (Hisab al-jabr wa'l-muqābalah)* was al-Khwārizmī's major work. This text on elementary practical mathematics contained sections on algebra, practical mensuration, and legacies. Our concern here is with the first section. His book was not only the first in Arabic to treat algebra but also gave the subject its name. The word *al-jabr* in the title meant "restoration" or "completion" and referred to the process of removing negative quantities, as when the expression $40 + x^2 = 18 + 10x$ is converted to $22 + x^2 = 10x$. Al-Khwārizmī solved problems that could be reduced to six standard forms of linear and quadratic equations. He divided these six forms into two groups of three types; in modern notation they are as follows:

Group 1	Group 2
$ax^2 = bx$	$ax^2 + bx = c$
$ax^2 = b$	$ax^2 + c = bx$
$ax = b$	$ax^2 = bx + c$, where a, b, and c are positive whole numbers.

These forms show that he recognized neither negative numbers nor zero as a coefficient. His *Algebra* suffered in that he used no symbols but expressed everything in words.

Al-Khwārizmī contributed to areas of mathematics besides algebra. His astronomical treatise contained an early sine table with base 150 (a common Hindu parameter). Though elementary, his treatise on Hindu numerals was important because it first systematically expounded the decimal system with digits 1 to 9, 0, and place value. His treatise introduced those numerals to Islamic works from which they were put into Latin translation in 12th-century Europe. Because European scholars linked al-Khwārizmī to the "new arithmetic" involving Hindu numerals, any treatise dealing with that topic was given the Latin form of his name, *algorismus*, a form subsequently corrupted to the modern algorithm.

47. From *The Book of Algebra and Almucabola**

(Quadratic Equations in Algebra: Verbal Form)

AL-KHWĀRIZMĪ

CONTAINING DEMONSTRATIONS OF THE RULES OF THE EQUATIONS OF ALGEBRA[1]

. . . Furthermore I discovered that the numbers of restoration and opposition are composed of these three kinds: namely, roots, squares, and numbers.[2] However, number alone is connected neither with roots nor with squares by any ratio. Of these, then, the root is anything composed of units which can be multiplied by itself, or any number greater than unity multiplied by itself: or that which is found to be diminished below unity when multiplied by itself. The square is that which results from the multiplication of a root by itself.

Of these three forms, then, two may be equal to each other, as for example:

Squares equal to roots,
Squares equal to numbers, and
Roots equal to numbers.[3]

CHAPTER I. CONCERNING SQUARES EQUAL TO ROOTS[4]

The following is an example of squares equal to roots: a square is equal to 5 roots. The root of the square then is 5, and 25 forms its square which, of course, equals five of its roots.

Another example: the third part of a square equals four roots. Then the root of the square is 12 and 144 designates its square. And similarly, five squares equal 10 roots. Therefore one square equals two roots and the root of the square is 2. Four represents the square.

In the same manner, then, that which

*Source: The English translation is originally from L. C. Karpinski's *Robert of Chester's Latin translation of the Algebra of Khowarizmi* (1915). It appeared with added notes in D. J. Struik (ed.), *A Source Book in Mathematics, 1200-1800* (1969), 56-60, and is reprinted by permission of Harvard University Press.

involves more than one square, or is less than one, is reduced to one square. Likewise you perform the same operation upon the roots which accompany the squares.

CHAPTER II. CONCERNING SQUARES EQUAL TO NUMBERS

Squares equal to numbers are illustrated in the following manner: a square is equal to nine. Then nine measures the square of which three represents one root.

Whether there are many or few squares, they will have to be reduced in the same manner to the form of one square. That is to say, if there are two or three or four squares, or even more, the equation formed by them with their roots is to be reduced to the form of one square with its root. Further, if there be less than one square, that is, if a third or a fourth or a fifth part of a square or root is proposed, this is treated in the same manner.

For example, five squares equal 80. Therefore one square equals the fifth part of the number 80 which, of course, is 16. Or, to take another example, half of a square equals 18. This square therefore equals 36. In like manner all squares, however many, are reduced to one square, or what is less than one is reduced to one square. The same operation must be performed upon the numbers which accompany the squares.

CHAPTER III. CONCERNING ROOTS EQUAL TO NUMBERS

The following is an example of roots equal to numbers: a root is equal to 3. Therefore nine is the square of this root.

Another example: four roots equal 20. Therefore one root of this square is 5. Still another example: half a root is equal to ten. The whole root therefore equals 20, of which, of course, 400 represents the square.

Therefore roots and squares and pure numbers are, as we have shown, distinguished from one another. Whence also from these three kinds which we have just explained, three distinct types of equations are formed involving three elements, as

A square and roots equal to numbers,
A square and numbers equal to roots, and
Roots and numbers equal to a square.

CHAPTER IV. CONCERNING SQUARES AND ROOTS EQUAL TO NUMBERS

The following is an example of squares and roots equal to numbers: a square and 10 roots are equal to 39 units.[5] The question therefore in this type of equation is about as follows: what is the square which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. Now the roots in the problem before us are 10. Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39, giving 64. Having taken then the square root of this which is 8, subtract from it the half of the roots, 5, leaving 3. The number three therefore represents one root of this square, which itself, of course, is 9. Nine therefore gives that square.

Similarly, however many squares are proposed all are to be reduced to one square. Similarly also you may reduce whatever numbers or roots accompany them in the same way in which you have reduced the squares.

The following is an example of this reduction: two squares and ten roots equal 48 units. The question therefore in this type of equation is something like this: what are the two squares which when combined are such that if ten roots of them are added, the sum total equals 48? First of all it is necessary that the two squares be reduced to one. But since one square is the half of two, it is at once evident that you should divide by two all the given terms in this problem. This gives a square and 5 roots equal to 24 units. The meaning of this is about as follows: what is the square which amounts to 24 when you add to it 5 of its roots? At the outset it is necessary, recalling the rule above given, that you take one-half of the roots. This gives two and one-half which multiplied by itself gives 6¼. Add this to 24, giving 30¼. Take then of this total the square root, which is, of course, 5½. From this subtract half of the roots, 2½, leaving 3, which expresses one root of the square, which itself is 9.

. . . .

CHAPTER VI. GEOMETRICAL DEMONSTRATIONS[6]

We have said enough, says al-Khwārizmī, so far as numbers are concerned, about the six types of equations. Now, however, it is necessary that we should demonstrate geometrically the truth of the same problems which we have explained in numbers. Therefore our first proposition is this, that a square and 10 roots equal 39 units.

The proof is that we construct [Fig. 1] a square of unknown sides, and let this square figure represent the square (second power of the unknown) which together with its root you wish to find. Let the square, then, be *ab*, of which any side represents one root. When we multiply any side of this by a number (or numbers) it is evident that that which results from the multiplication will be a number of roots equal to the root of the same number (of the square). Since then ten roots were proposed with the square, we take a fourth part of the number ten and apply to each side of the square an area of equidistant sides, of which the length should be the same as the length of the square first described and the breadth 2½, which is a fourth part of 10. Therefore four areas of equidistant sides are applied to the first square, *ab*. Of each of these the length is the length of one root of the square *ab* and also the breadth of each is 2½, as we have just said. These now are the areas *c, d, e, f*. Therefore it follows from what we have said that there will be four areas having sides of unequal length, which also are regarded as unknown. The size of the areas in each of the four corners, which is found by multiplying 2½ by 2½, completes that which is lacking in the larger or whole area. Whence it is we complete the drawing of the larger area by the addition of the four products, each 2½ by 2½; the whole of this multiplication gives 25.

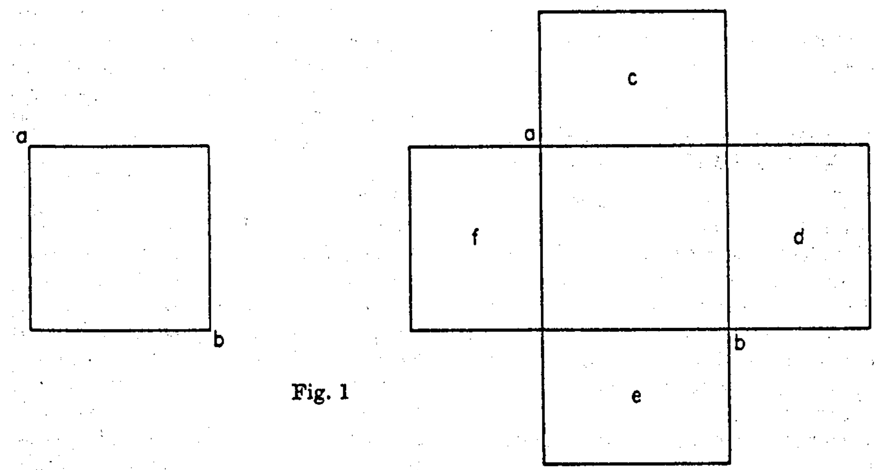

Fig. 1

And now it is evident that the first square figure, which represents the square of the unknown [x^2], and the four surrounding areas [$10x$] make 39. When we add 25 to this, that is, the four smaller squares which indeed are placed at the four angles of the square *ab*, the drawing of the larger square, called *GH*, is completed [Fig. 2]. Whence also the sum total of this is 64, of which 8 is the root, and by this is designated one side of the completed figure. Therefore when we subtract from eight twice the fourth part of 10, which is placed at the extremities of the larger square *GH*, there will remain but 3. Five being subtracted from 8, 3 necessarily remains, which is equal to one side of the first square *ab*.

This three then expresses one root of the square figure, that is, one root of the proposed square of the unknown, and 9 the square itself. Hence we take half of ten and multiply this by itself. We then add the whole product of the multiplication to 39, that the drawing of the larger square *GH* may be completed; for the lack of the four corners rendered incomplete the drawing of the whole of this square. Now it is evident that the fourth part of any number multiplied by itself and then multiplied by four gives the same number as half of the number multiplied by itself. Therefore if half of the root is multiplied by itself, the sum total of this multiplication will wipe out, equal, or cancel the multiplication of the fourth part by itself and then by four.

The remainder of the treatise deals with problems that can be reduced to one of the six types, for example, how to divide 10 into two parts in such a way that the sum of the products obtained by multiplying each part by itself is equal to 58: $x^2 + (10 - x)^2 = 58, x = 3, x = 7$. This is followed by a section on problems of inheritance.

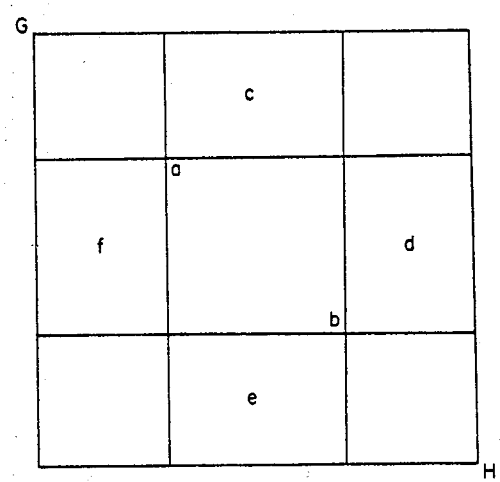

Fig. 2

NOTES

1. *Jabr* is the setting of a bone, hence reduction or restoration; *muqabala* is confrontation, opposition, face-to-face (explanation by Professor E. S. Kennedy).

2. The term "roots" *(radices)* stands for multiples of the unknown, our x; the term "squares" *(substantiae)* stands for multiples of our x^2; "numbers" *(numeri)* are constants.

3. In our notation, $x^2 = ax$, $x^2 = b$, $x = c$.

4. Latin: *de substantiis numeros coaequantibus*. The examples are $x^2 = 5x$, $\frac{1}{3}x^2 = 4x$, $5x^2 = 10x$.

5. This example, $x^2 + 10x = 39$, answer $x = 3$, "runs," as Karpinski notices in his introduction to this translation, "like a thread of gold through the algebras for several centuries, appearing in the algebras of Abu Kamil, Al-Karkhi and Omar al-Khayyami, and frequently in the works of Christian writers," and it still graces our present algebra texts. The solution of this type, $x^2 + ax = b$, is, as we can verify, based on the formula $x = \sqrt{(a/2)^2 + b} - a/2$.

6. For these geometric demonstrations we must go back, as said, to Euclid's *Elements* (Book VI, Prop. 28, 29; see also Book II, Prop. 5, 6). See also on this subject the introduction to the *Principal works of Simon Stevin*, vol. IIB (Swets-Zeitlinger, Amsterdam, 1958). . . .