Definition. Let $p : E \to B$ be a continuous surjective map. The open set U of B is said to be ***evenly covered*** by p if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in E such that for each α, the restriction of p to V_α is a homeomorphism of V_α onto U. The collection $\{V_\alpha\}$ will be called a partition of $p^{-1}(U)$ into ***slices***.

If U is an open set that is evenly covered by p, we often picture the set $p^{-1}(U)$ as a "stack of pancakes," each having the same size and shape as U, floating in the air above U; the map p squashes them all down onto U. See Figure 53.1. Note that if U is evenly covered by p and W is an open set contained in U, then W is also evenly covered by p.

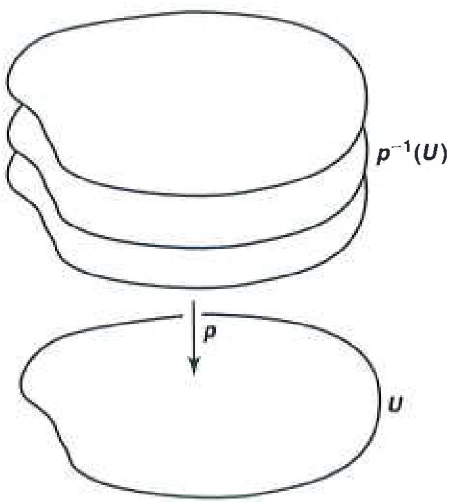

Figure 53.1

Definition. Let $p : E \to B$ be continuous and surjective. If every point b of B has a neighborhood U that is evenly covered by p, then p is called a ***covering map***, and E is said to be a ***covering space*** of B.

Note that if $p : E \to B$ is a covering map, then for each $b \in B$ the subspace $p^{-1}(b)$ of E has the discrete topology. For each slice V_α is open in E and intersects the set $p^{-1}(b)$ in a single point; therefore, this point is open in $p^{-1}(b)$.

Note also that if $p : E \to B$ is a covering map, then p is an open map. For suppose A is an open set of E. Given $x \in p(A)$, choose a neighborhood U of x that is evenly covered by p. Let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices. There is a point y of A such that $p(y) = x$; let V_β be the slice containing y. The set $V_\beta \cap A$ is open in E and hence open in V_β; because p maps V_β homeomorphically onto U, the set $p(V_\beta \cap A)$ is open in U and hence open in B; it is thus a neighborhood of x contained in $p(A)$, as desired.

EXAMPLE 1. Let X be any space; let $i : X \to X$ be the identity map. Then i is a covering map (of the most trivial sort). More generally, let E be the space $X \times \{1, \dots, n\}$ consisting of n disjoint copies of X. The map $p : E \to X$ given by $p(x, i) = x$ for all i is again a (rather trivial) covering map. In this case, we can picture the entire space E as a stack of pancakes over X.

In practice, one often restricts oneself to covering spaces that are path connected, to eliminate trivial coverings of the pancake-stack variety. An example of such a nontrivial covering space is the following:

Theorem 53.1. *The map $p : \mathbb{R} \to S^1$ given by the equation*

$$p(x) = (\cos 2\pi x, \sin 2\pi x)$$

is a covering map.

One can picture p as a function that wraps the real line $\mathbb{R}$ around the circle S^1, and in the process maps each interval $[n, n+1]$ onto S^1.

Proof. The fact that p is a covering map comes from elementary properties of the sine and cosine functions. Consider, for example, the subset U of S^1 consisting of those points having positive first coordinate. The set $p^{-1}(U)$ consists of those points x for which $\cos 2\pi x$ is positive; that is, it is the union of the intervals

$$V_n = (n - \tfrac{1}{4}, n + \tfrac{1}{4}),$$

for all $n \in \mathbb{Z}$. See Figure 53.2. Now, restricted to any closed interval $\bar{V}_n$, the map p is injective because $\sin 2\pi x$ is strictly monotonic on such an interval. Furthermore, p carries $\bar{V}_n$ surjectively onto $\bar{U}$, and V_n to U, by the intermediate value theorem. Since $\bar{V}_n$ is compact, $p|\bar{V}_n$ is a homeomorphism of $\bar{V}_n$ with $\bar{U}$. In particular, $p|V_n$ is a homeomorphism of V_n with U.

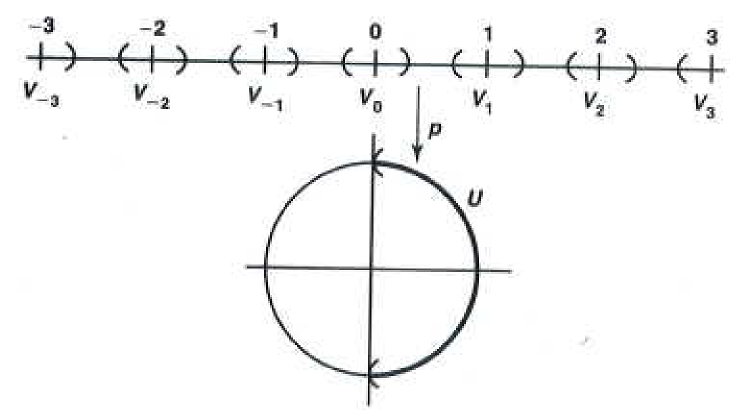

Figure 53.2

Similar arguments can be applied to the intersections of S^1 with the upper and lower open half-planes, and with the open left-hand half-plane. These open sets cover S^1, and each of them is evenly covered by p. Hence $p : \mathbb{R} \to S^1$ is a covering map. ∎

If $p : E \to B$ is a covering map, then p is a ***local homeomorphism*** of E with B. That is, each point e of E has a neighborhood that is mapped homeomorphically by p onto an open subset of B. The condition that p be a local homeomorphism does not suffice, however, to ensure that p is a covering map, as the following example shows.

EXAMPLE 2. The map $p : \mathbb{R}_+ \to S^1$ given by the equation

$$p(x) = (\cos 2\pi x, \sin 2\pi x)$$

is surjective, and it is a local homeomorphism. See Figure 53.3. But it is not a covering map, for the point $b_0 = (1, 0)$ has no neighborhood U that is evenly covered by p. The typical neighborhood U of b_0 has an inverse image consisting of small neighborhoods V_n of each integer n for $n > 0$, along with a small interval V_0 of the form $(0, \epsilon)$. Each of the intervals V_n for $n > 0$ is mapped homeomorphically onto U by the map p, but the interval V_0 is only imbedded in U by p.

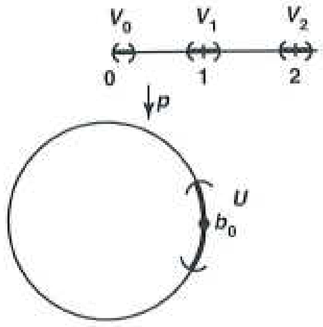

Figure 53.3

EXAMPLE 3. The preceding example might lead you to think that the real line $\mathbb{R}$ is the only connected covering space of the circle S^1. This is not so. Consider, for example, the map $p : S^1 \to S^1$ given in equations by

$$p(z) = z^2.$$

[Here we consider S^1 as the subset of the complex plane $\mathbb{C}$ consisting of those complex numbers z with $|z| = 1$.] We leave it to you to check that p is a covering map.

Example 2 shows that the map obtained by restricting a covering map may not be a covering map. Here is one situation where it *will* be a covering map:

Theorem 53.2. *Let $p : E \to B$ be a covering map. If B_0 is a subspace of B, and if $E_0 = p^{-1}(B_0)$, then the map $p_0 : E_0 \to B_0$ obtained by restricting p is a covering map.*

Proof. Given $b_0 \in B_0$, let U be an open set in B containing b_0 that is evenly covered by p; let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices. Then $U \cap B_0$ is a neighborhood of b_0 in B_0, and the sets $V_\alpha \cap E_0$ are disjoint open sets in E_0 whose union is $p^{-1}(U \cap B_0)$, and each is mapped homeomorphically onto $U \cap B_0$ by p. ∎

Theorem 53.3. *If $p : E \to B$ and $p' : E' \to B'$ are covering maps, then*

$$p \times p' : E \times E' \to B \times B'$$

is a covering map.

Proof. Given $b \in B$ and $b' \in B'$, let U and U' be neighborhoods of b and b', respectively, that are evenly covered by p and p', respectively. Let $\{V_\alpha\}$ and $\{V'_\beta\}$ be partitions of $p^{-1}(U)$ and $(p')^{-1}(U')$, respectively, into slices. Then the inverse image under $p \times p'$ of the open set $U \times U'$ is the union of all the sets $V_\alpha \times V'_\beta$. These are disjoint open sets of $E \times E'$, and each is mapped homeomorphically onto $U \times U'$ by $p \times p'$. ∎

EXAMPLE 4. Consider the space $T = S^1 \times S^1$; it is called the ***torus***. The product map

$$p \times p : \mathbb{R} \times \mathbb{R} \longrightarrow S^1 \times S^1$$

is a covering of the torus by the plane $\mathbb{R}^2$, where p denotes the covering map of Theorem 53.1. Each of the unit squares $[n, n+1] \times [m, m+1]$ gets wrapped by $p \times p$ entirely around the torus. See Figure 53.4.

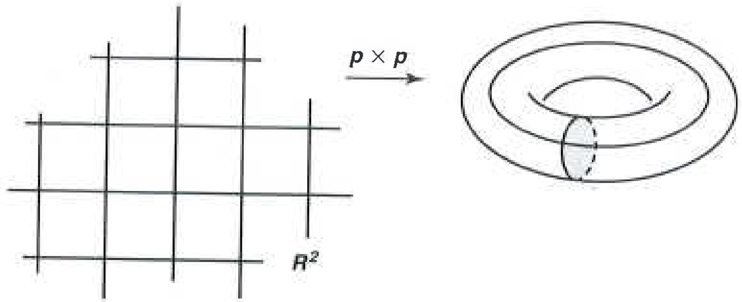

Figure 53.4

In this figure, we have pictured the torus not as the product $S^1 \times S^1$, which is a subspace of $\mathbb{R}^4$ and thus difficult to visualize, but as the familiar doughnut-shaped surface D in $\mathbb{R}^3$ obtained by rotating the circle C_1 in the xz-plane of radius $\frac{1}{3}$ centered at $(1, 0, 0)$ about the z-axis. It is not hard to see that $S^1 \times S^1$ is homeomorphic with the surface D. Let C_2 be the circle of radius 1 in the xy-plane centered at the origin. Then let us map $C_1 \times C_2$ into D by defining $f(a \times b)$ to be that point into which a is carried when one rotates the circle C_1 about the z-axis until its center hits the point b. See Figure 53.5. The map f will be a homeomorphism of $C_1 \times C_2$ with D, as you can check mentally. If you wish, you can write equations for f and check continuity, injectivity, and surjectivity directly. (Continuity of f^{-1} will follow from compactness of $C_1 \times C_2$.)

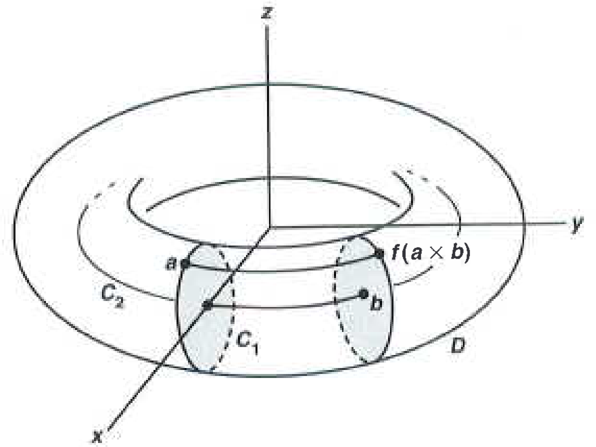

Figure 53.5

EXAMPLE 5. Consider the covering map $p \times p$ of the preceding example. Let b_0 denote the point $p(0)$ of S^1; and let B_0 denote the subspace

$$B_0 = (S^1 \times b_0) \cup (b_0 \times S^1)$$

of $S^1 \times S^1$. Then B_0 is the union of two circles that have a point in common; we sometimes call it the ***figure-eight space***. The space $E_0 = p^{-1}(B_0)$ is the "infinite grid"

$$E_0 = (\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R})$$

pictured in Figure 53.4. The map $p_0 : E_0 \to B_0$ obtained by restricting $p \times p$ is thus a covering map.

The infinite grid is but one covering space of the figure eight; we shall see others later on.

EXAMPLE 6. Consider the covering map

$$p \times i : \mathbb{R} \times \mathbb{R}_+ \longrightarrow S^1 \times \mathbb{R}_+,$$

where i is the identity map of $\mathbb{R}_+$ and p is the map of Theorem 53.1. If we take the standard homeomorphism of $S^1 \times \mathbb{R}_+$ with $\mathbb{R}^2 - \mathbf{0}$, sending $x \times t$ to tx, the composite gives us a covering

$$\mathbb{R} \times \mathbb{R}_+ \longrightarrow \mathbb{R}^2 - \mathbf{0}$$

of the punctured plane by the open upper half-plane. It is pictured in Figure 53.6. This covering map appears in the study of complex variables as the *Riemann surface* corresponding to the complex logarithm function.

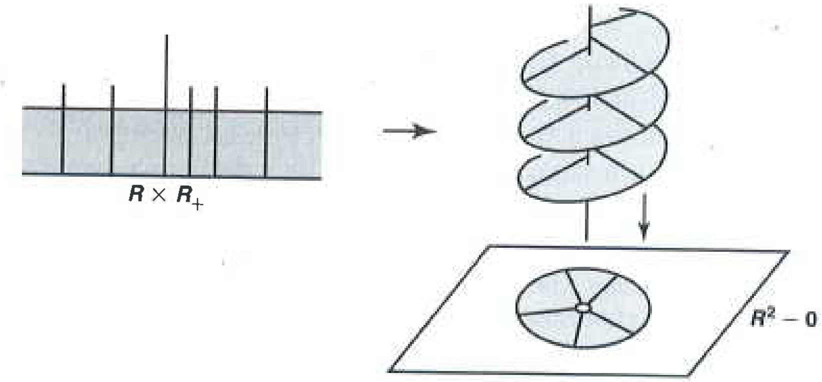

Figure 53.6

Exercises

1. Let Y have the discrete topology. Show that if $p : X \times Y \to X$ is projection on the first coordinate, then p is a covering map.
2. Let $p : E \to B$ be continuous and surjective. Suppose that U is an open set of B that is evenly covered by p. Show that if U is connected, then the partition of $p^{-1}(U)$ into slices is unique.
3. Let $p : E \to B$ be a covering map; let B be connected. Show that if $p^{-1}(b_0)$ has k elements for some $b_0 \in B$, then $p^{-1}(b)$ has k elements for every $b \in B$. In such a case, E is called a ***k-fold covering*** of B.
4. Let $q : X \to Y$ and $r : Y \to Z$ be covering maps; let $p = r \circ q$. Show that if $r^{-1}(z)$ is finite for each $z \in Z$, then p is a covering map.
5. Show that the map of Example 3 is a covering map. Generalize to the map $p(z) = z^n$.
6. Let $p : E \to B$ be a covering map.
   (a) If B is Hausdorff, regular, completely regular, or locally compact Hausdorff, then so is E. [*Hint:* If $\{V_\alpha\}$ is a partition of $p^{-1}(U)$ into slices, and C is a closed set of B such that $C \subset U$, then $p^{-1}(C) \cap V_\alpha$ is a closed set of E.]
   (b) If B is compact and $p^{-1}(b)$ is finite for each $b \in B$, then E is compact.

§54 The Fundamental Group of the Circle

The study of covering spaces of a space X is intimately related to the study of the fundamental group of X. In this section, we establish the crucial links between the two concepts, and compute the fundamental group of the circle.

CHAPTER V

Covering Spaces

§1. Introduction

Let X be a topological space: *a covering space of* X consists of a space $\tilde{X}$ and a continuous map p of $\tilde{X}$ onto X which satisfies a certain very strong smoothness requirement. The precise definition is given below. The theory of covering spaces is important not only in topology, but also in related disciplines such as differential geometry, the theory of Lie groups, and the theory of Riemann surfaces.

The theory of covering spaces is closely connected with the study of the fundamental group. Many basic topological questions about covering spaces can be reduced to purely algebraic questions about the fundamental groups of the various spaces involved. It would be practically impossible to give a complete exposition of either one of these two topics without also taking up the other.

§2. Definition and Some Examples of Covering Spaces

In this chapter, we shall assume that *all spaces are arcwise connected and locally arcwise connected* (see §II.2 for the definition) unless otherwise stated. To save words, we shall not keep repeating this assumption. On the other hand, it is not necessary to assume that the spaces we are dealing with satisfy any separation axioms.

Definition. Let X be a topological space. A *covering space* of X is a pair consisting of a space $\tilde{X}$ and a continuous map $p : \tilde{X} \to X$ such that the following condition holds: Each point $x \in X$ has an arcwise-connected open neighborhood U such that each arc component of $p^{-1}(U)$ is mapped topologically onto U by p [in particular, it is assumed that $p^{-1}(U)$ is nonempty]. Any open neighborhood U that satisfies the condition just stated is called an *elementary neighborhood*. The map p is often called a *projection*.

To clarify this definition, we now give several examples. In some of the examples our discussion will be rather informal, which is often more helpful than a more rigorous and formal discussion in getting an intuitive feeling for the concept of covering space.

Examples

2.1. Let $p : \mathbf{R} \to S^1$ be defined by

$$p(t) = (\sin t, \cos t)$$

for any $t \in \mathbf{R}$. Then, the pair $(\mathbf{R}, p)$ is a covering space of the unit circle S^1. Any open subinterval of the circle S^1 can be serve as an elementary neighborhood. This is one of the simplest and most important examples.

2.2. Let us use polar coordinates (r, θ) in the plane $\mathbf{R}^2$. Then, the unit circle S^1 is defined by the condition $r = 1$. For any integer n, positive or negative, define a map $p_n : S^1 \to S^1$ by the equation

$$p_n(1, \theta) = (1, n\theta).$$

The map p_n wraps the circle around itself n times. It is readily seen that, if $n \neq 0$, the pair (S^1, p_n) is a covering space of S^1. Once again, any proper open interval in S^1 is an elementary neighborhood.

2.3. If X is any space, and $i : X \to X$ denotes the identity map, then the pair (X, i) is a trivial example of a covering space of X. Similarly, if f is a homeomorphism of Y onto X, then (Y, f) is a covering space of X, which is also a rather trivial example. Later in this chapter, we shall prove that, if X is simply connected, then any covering space of X is one of these trivial covering spaces. Thus, we can only hope for nontrivial examples of covering spaces in the case of spaces that are not simply connected.

2.4. If $(\tilde{X}, p)$ is a covering space of X, and $(\tilde{Y}, q)$ is a covering space of Y, then $(\tilde{X} \times \tilde{Y}, p \times q)$ is a covering space of $X \times Y$ [the map $p \times q$ is defined by $(p \times q)(x, y) = (px, qy)$]. We leave the proof to the reader. It is clear that, if U is an elementary neighborhood of the point $x \in X$ and V is an elementary neighborhood of the point $y \in Y$, then $U \times V$ is an elementary neighborhood of $(x, y) \in X \times Y$.

Using this result and Examples 2.1 and 2.2, the reader can construct examples of covering spaces of the torus $T = S^1 \times S^1$. In particular, the plane $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$, the cylinder $\mathbf{R} \times S^1$, or the torus itself can serve as a covering

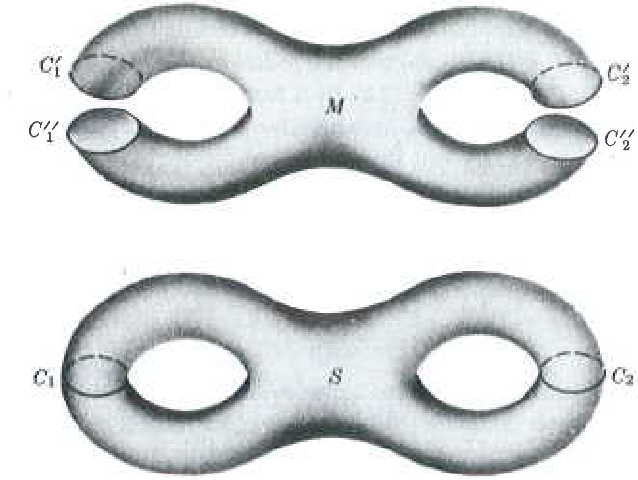

FIGURE 5.1. A surface of genus 2 as a quotient space of a bordered surface.

space of the torus. The reader should try to visualize the projection p involved in each of these cases.

2.5. In §I.4, the projective plane P was defined as a quotient space of the 2-sphere S^2. Let $p : S^2 \to P$ denote the natural map. Then, it is readily seen that (S^2, p) is a covering space of P. We can take as an elementary neighborhood of any point $x \in P$ an open disc containing x.

2.6. Let S be a compact, orientable surface of genus 2. We shall show how to construct a great variety of covering spaces of S. Note that we can regard S as a quotient space of a compact, bordered surface M, where M is orientable, of genus 0, and its boundary consists of four circles C_1', C_1'', C_2', and C_2''. The natural map $M \to S$ identifies the boundary circles in pairs (see Figure 5.1): C_i' and C_i'' are identified to a single circle C_i by means of a homeomorphism h_i of C_i' onto C_i'', $i = 1, 2$. We can also think of M as obtained from S by cutting along the circles C_1 and C_2.

Let D be the finite set $\{1, 2, 3, \ldots, n\}$ with the discrete topology and $q : M \times D \to M$, the projection of the product space onto the first factor. We can think of $M \times D$ as consisting of n disjoint copies of M, each of which is mapped homeomorphically onto M by q. We now describe how to form a quotient space of $M \times D$, which will be a connected 2-manifold $\tilde{S}$ and such that the map q will induce a map $p : \tilde{S} \to S$ of quotient spaces; i.e., so we will have a commutative diagram

$$\begin{array}{ccc} M \times D & \longrightarrow & \tilde{S} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ M & \longrightarrow & S \end{array}$$

It will turn out that $(\tilde{S}, p)$ is a covering space of S. The identification by which we form $\tilde{S}$ from $M \times D$ will all be of the following form: The circle $C_i' \times \{j\}$ is identified with the circle $C_i'' \times \{k\}$ by a homeomorphism which sends the point (x, j) onto the point $(h_i(x), k)$, where $i = 1$ or 2, and j and k are positive integers $\leq n$. We can carry out this identification of circles in pairs in many different ways, so long as we obtain a space $\tilde{S}$ which is connected. For example, in the case where $n = 3$, we could carry out the identifications according to the following scheme: Identify

$$\begin{aligned} C_1' \times \{1\} &\quad \text{with} \quad C_1'' \times \{2\},\\ C_1' \times \{2\} &\quad \text{with} \quad C_1'' \times \{3\},\\ C_1' \times \{3\} &\quad \text{with} \quad C_1'' \times \{1\},\\ C_2' \times \{1\} &\quad \text{with} \quad C_2'' \times \{2\},\\ C_2' \times \{2\} &\quad \text{with} \quad C_2'' \times \{1\},\\ C_2' \times \{3\} &\quad \text{with} \quad C_2'' \times \{3\}. \end{aligned}$$

We leave it to the reader to concoct other examples and to prove that in each case we actually obtain a covering space. Obviously, we could use a similar procedure to obtain examples of covering spaces of surfaces of higher genus.

2.7. Let X be a subset of the plane consisting of two circles tangent at a point:

$$\begin{aligned} C_1 &= \{(x, y) : (x - 1)^2 + y^2 = 1\},\\ C_2 &= \{(x, y) : (x + 1)^2 + y^2 = 1\},\\ X &= C_1 \cup C_2. \end{aligned}$$

We shall give two different examples of covering spaces of X. For the first example, let $\tilde{X}$ denote the set of all points $(x, y) \in \mathbf{R}^2$ such that x or y (or both) is an integer; $\tilde{X}$ is a union of horizontal and vertical straight lines. Define $p : \tilde{X} \to X$ by the formula

$$p(x, y) = \begin{cases} (1 + \cos(\pi - 2\pi x), \sin 2\pi x) & \text{if } y \text{ is an integer,}\\ (-1) + \cos 2\pi y, \sin 2\pi y) & \text{if } x \text{ is an integer.} \end{cases}$$

The map p wraps each horizontal line around the circle C_1 and each vertical line around the circle C_2.

For the second example, let D_n denote the circle $\{(x, y) \in \mathbf{R}^2 : (x - 1)^2 + (y - 3n)^2 = 1\}$ for any integer n, positive, negative, or zero, and let L denote the vertical line $\{(x, y) : x = 0\}$. The circles D_n are pairwise disjoint, and each is tangent to the line L. Define

$$\tilde{X}' = L \cup \left(\bigcup_{n \in Z} D_n \right),$$

and $p' : \tilde{X}' \to X$ as follows: Let p' map each circle D_n homeomorphically onto

C_1 by a vertical translation of the proper amount. Let p' wrap the line L around the circle C_2 in accordance with the formula

$$p'(0, y) = \left(-1 + \cos \frac{2\pi y}{3}, \sin \frac{2\pi y}{3} \right).$$

Then, $(\tilde{X}', p')$ is a covering space of X.

2.8. Here is an example for students who have at least a slight familiarity with the theory of functions of a complex variable. As usual, let

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

denote the exponential function, where z is any complex number. The exponential function is a map, $\exp : \mathbf{C} \to \mathbf{C} - \{0\}$, where $\mathbf{C}$ denotes the complex plane. We assert that $(\mathbf{C}, \exp)$ is a covering space of $\mathbf{C} - \{0\}$, and that, for any $z \in \mathbf{C} - \{0\}$, the open disc

$$U_z = \{w \in \mathbf{C} : |w - z| < |z|\}$$

is an elementary neighborhood. To prove this, we would have to show that any component V of the inverse image of U_z is mapped homeomorphically onto U_z by exp; i.e., that there exists a continuous function $f : U_z \to V$ such that, for any $w \in U_z$,

$$\exp[f(w)] = w,$$

and, for any $v \in V$,

$$f(\exp v) = v.$$

Such a function f is called a "branch of the logarithm function in the disc U_z" in books on complex variables, and in the course of establishing the properties of the logarithm, the required facts are proved.

Recall that, if $z = x + iy$, then $\exp z = (\exp x)\cdot(\cos y + i \sin y)$, where $\exp x = e^x$ now refers to the more familiar real exponential function, $\exp : \mathbf{R} \to \{t \in \mathbf{R} : t > 0\}$. From this formula, the following fact emerges. We can regard $\mathbf{C} = \mathbf{R} \times \mathbf{R}$ and $\mathbf{C} - \{0\} = \{r \in \mathbf{R} : r > 0\} \times S^1$ (use polar coordinates). Then, we can consider the map $\exp : \mathbf{C} \to \mathbf{C} - \{0\}$ as a map $p \times q : \mathbf{R} \times \mathbf{R} \to \{r \in \mathbf{R} : r > 0\} \times S^1$, where $p(x) = e^x$ and $q(y) = (\cos y, \sin y)$. Compare Examples 2.1, 2.3, and 2.4.

2.9. We now give another example from the theory of functions of a complex variable. For any integer $n \neq 0$, let $p_n : \mathbf{C} \to \mathbf{C}$ be defined by $p_n(z) = z^n$. Then, $(\mathbf{C} - \{0\}, p_n)$ is a covering space of $\mathbf{C} - \{0\}$. The proof is given in books on complex variables when the existence and properties of the various "branches" of the function $\sqrt[n]{z}$ are discussed; the situation is analogous to that in Example 2.8. Note that it is necessary to omit 0 from the domain and range of the function p_n; otherwise we would not have a covering space. As in Example 2.8, we can consider $\mathbf{C} - \{0\} = \{r \in \mathbf{R} : r > 0\} \times S^1$ and decompose the covering space $(\mathbf{C} - \{0\}, p_n)$ into the Cartesian product of two covering spaces.

To clarify further the concept of covering space, we shall give some examples which are almost, but not quite, covering spaces.

Definition. A continuous map $f: X \to Y$ is a *local homeomorphism* if each point $x \in X$ has an open neighborhood V such that $f(V)$ is open and f maps V topologically onto $f(V)$.

It is readily proved that, if $(\tilde{X}, p)$ is a covering space of X, then p is a local homeomorphism (the proof depends on the fact that in a locally arcwise connected space, the arc components of an open set are open). Also, the inclusion map of an open subset of a toplogical space into the whole space is a local homeomorphism. Finally, the composition of two local homeomorphisms is again a local homeomorphism. Thus, we can construct many examples of local homeomorphisms.

On the other hand, it is easy to construct examples of local homeomorphisms which are onto maps, but not covering spaces. For example, let p map the open interval $(0, 10)$ onto the circle S^1 as follows:

$$p(t) = (\cos t, \sin t).$$

Then, p is a local homeomorphism, but $((0, 10), p)$ is not a covering space of S^1. (Which points of S^1 fail to have an elementary neighborhood?) More generally, if $(\tilde{X}, p)$ is a covering space of X, and V is a connected, open, proper subset of $\tilde{X}$, then $p|V$ is a local homeomorphism, but $(V, p|V)$ is not a covering space of X. It is important to keep this distinction between covering spaces and local homeomorphisms in mind.

Note that a local homeomorphism is an open map. In particular, if $(\tilde{X}, p)$ is a covering space of X, then p is an open map.

We next give a lemma which makes it possible to give many additional examples of covering spaces.

Lemma 2.1. *Let $(\tilde{X}, p)$ be a covering space of X, let A be a subspace of X which is arcwise connected and locally arcwise connected, and let $\tilde{A}$ be an arc component of $p^{-1}(A)$. Then, $(\tilde{A}, p|\tilde{A})$ is a covering space of A.*

The proof is immediate. The two covering spaces described in Example 2.7 can also be obtained by applying this lemma to the covering spaces $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ and $\mathbf{R} \times S^1$ of the torus $S^1 \times S^1$ described in Example 2.4 [choose A to be the following subset of $S^1 \times S^1$: $A = (S^1 \times \{x_0\}) \cup (\{x_0\} \times S^1)$, where $x_0 \in S^1$].

We close this section by stating two of the principal problems in the theory of covering spaces:

(a) Give necessary and sufficient conditions for two covering spaces $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ of X to be isomorphic (by definition, they are

isomorphic if and only if there exists a homeomorphism h of $\tilde{X}_1$ onto $\tilde{X}_2$ such that $p_2 h = p_1$).

(b) Given a space X, determine all possible covering spaces of X (up to isomorphism).

As we shall see, these problems have reasonable answers in terms of the fundamental groups of the spaces involved.

Exercises

2.1. Prove that the following four conditions on a topological space are equivalent:

(a) They are components of any open subset are open.
(b) Every point has a basic family of arcwise-connected open neighborhoods.
(c) Every point has a basic family of arcwise-connected neighborhoods (they are not assumed to be open).
(d) For every point x and every neighborhood U of x, there exists a neighborhood V of x such that $V \subset U$ and any two points of V can be joined by an arc in U.

Thus, any one of these conditions could be taken as the definition of local arcwise connectivity.

2.2. Give an example of a local homeomorphism $f: X \to Y$ and a subset $A \subset X$ such that $f|A$ is not a local homeomorphism of A onto $f(A)$.

2.3. Prove that if X is compact and $f: X \to Y$ is a local homeomorphism, then, for any point $y \in Y$, $f^{-1}(y)$ is a finite set. If it is also assumed that Y is a connected Hausdorff space, then f maps X onto Y.

2.4. Assume X and Y are arcwise connected and locally arcwise connected, X is compact Hausdorff, and Y is Hausdorff. Let $f: X \to Y$ be a local homeomorphism; prove that (X, f) is a covering space of Y. (Warning: This exercise is more subtle than it looks!)

§3. Lifting of Paths to a Covering Space

In this section, we prove some simple lemmas which provide the key to many of the results in this chapter. Let $(\tilde{X}, p)$ be a covering space of X, and let $g: I \to \tilde{X}$ be a path in $\tilde{X}$; then, pg is a path in X. Also, if $g_0, g_1: I \to \tilde{X}$ and $g_0 \sim g_1$, then $pg_0 \sim pg_1$. We can now ask for a sort of converse result: If $f: I \to X$ is a path in X, does there exist a path $g: I \to \tilde{X}$ such that $pg = f$? If $g_0, g_1: I \to \tilde{X}$ and $pg_0 \sim pg_1$, does it follow that $g_0 \sim g_1$? We shall see that the answer to both questions is *Yes*. This fact expresses one of the basic properties of covering spaces.

Lemma 3.1. *Let $(\tilde{X}, p)$ be a covering space of X, $\tilde{x}_0 \in \tilde{X}$, and $x_0 = p(\tilde{x}_0)$. Then, for any path $f: I \to X$ with initial point x_0, there exists a unique path $g: I \to \tilde{X}$ with initial point $\tilde{x}_0$ such that $pg = f$.*

PROOF. If the path f were contained in an elementary neighborhood U there would be no problem. For, if V denotes the arc component of $p^{-1}(U)$ which contains $\tilde{x}_0$, then, because p maps V topologically onto U, there would exist a unique g in V with the required properties.

Of course, f will not, in general, be contained in an elementary neighborhood U. However, we can always express f as the product of a finite number of "shorter" paths, each of which is contained in an elementary neighborhood, and then apply the argument in the preceding paragraph to each of these shorter paths in succession.

The details of this procedure may be described as follows. Let $\{U_i\}$ be a covering of X by elementary neighborhoods; then $\{f^{-1}(U_i)\}$ *is an open covering of the compact metric space I*. Choose an integer n so large that $1/n$ is less than the Lebesgue number of this covering. Divide the interval I into the closed subintervals $[0, 1/n]$, $[1/n, 2/n]$, $\ldots$, $[(n-1)/n, 1]$. Note that f maps each subinterval into an elementary neighborhood in X. We now define g successively over these subintervals, starting with $[0, 1/n]$.

The uniqueness of the lifted path g is a consequence of the following more general lemma.

Lemma 3.2. *Let $(\tilde{X}, p)$ be a covering space of X and let Y be a space which is connected. Given any two continuous maps $f_0, f_1 : Y \to \tilde{X}$ such that $pf_0 = pf_1$, the set $\{y \in Y : f_0(y) = f_1(y)\}$ is either empty or all of Y.*

PROOF. Because Y is connected, it suffices to prove that the set in question is both open and closed. First we shall prove that it is closed. Let y be a point of the closure of this set, and let

$$x = pf_0(y) = pf_1(y).$$

Assume $f_0(y) \neq f_1(y)$; we will show that this assumption leads to a contradiction. Let U be an elementary neighborhood of x, and let V_0 and V_1 be the components of $p^{-1}(U)$ which contain $f_0(y)$ and $f_1(y)$, respectively. Since f_0 and f_1 are both continuous, we can find a neghborhood W of y such that $f_0(W) \subset V_0$ and $f_1(W) \subset V_1$. But it is readily seen that this contradicts the fact that any neighborhood W of y must meet the set in question.

An analogous argument enables us to show that every point of the set $\{y \in Y : f_0(y) = f_1(y)\}$ is an interior point. Q.E.D.

Lemma 3.3. *Let $(\tilde{X}, p)$ be a covering space of X and let $g_0, g_1 : I \to \tilde{X}$ be paths in $\tilde{X}$ which have the same initial point. If $pg_0 \sim pg_1$, then $g_0 \sim g_1$; in particular, g_0 and g_1 have the same terminal point.*

PROOF. The strategy of this proof is essentially the same as that of Lemma 3.1. Let $\tilde{x}_0$ be the initial point of g_0 and g_1. The hypothesis $pg_0 \sim pg_1$ implies the existence of a map $F : I \times I \to X$ such that

$$F(s, 0) = pg_0(s),$$

$$F(s, 1) = pg_1(s),$$

$$F(0, t) = pg_0(0) = p(\tilde{x}_0),$$

$$F(1, t) = pg_0(1).$$

By an argument using the Lebesgue number, etc., we can find numbers $0 = s_0 < s_1 < \cdots < s_m = 1$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ such that F maps each small rectangle $[s_{i-1}, s_i] \times [t_{j-1}, t_j]$ into some elementary neighborhood in X. We shall prove that there exists a unique map $G : I \times I \to \tilde{X}$ such that $pG = F$ and $G(0, 0) = \tilde{x}_0$. First, we define G over the small rectangle $[0, s_1] \times [0, t_1]$ so that the required properties hold; it is clear that this can be done because F maps this small rectangle into an elementary neighborhood of the point $p(\tilde{x}_0)$. Then, we extend the definition of G successively over the rectangles $[s_{i-1}, s_i] \times [0, t_1]$ for $i = 2, 3, \ldots, m$, taking care that the definitions agree on the common edge of any two successive rectangles. Thus, G is defined over the strip $I \times [0, t_1]$. Next, G is defined over the rectangles in the strip $I \times [t_1, t_2]$, etc.

The uniqueness of G is assured by Lemma 3.2. Similarly, by the uniqueness assertion of Lemma 3.1, we see that $G(s, 0) = g_0(s)$, $G(0, t) = \tilde{x}_0$, $G(s, 1) = g_1(s)$, and that G maps $\{1\} \times I$ into a single point $\tilde{x}_1$ such that

$$p(\tilde{x}_1) = pg_0(1) = pg_1(1).$$

Thus, G defines an equivalence between the paths g_0 and g_1 as required. Q.E.D.

As a corollary to these results on the lifting of paths, we shall prove the following lemma:

Lemma 3.4. *If $(\tilde{X}, p)$ is a covering space of X, then the sets $p^{-1}(x)$ for all $x \in X$ have the same cardinal number.*

PROOF. Let x_0 and x_1 be any two points of X. Choose a path f in X with initial point x_0 and terminal point x_1. Using the path f, we can define a mapping $p^{-1}(x_0) \to p^{-1}(x_1)$ by the following procedure. Given any point $y_0 \in p^{-1}(x_0)$, lift f to a path g in $\tilde{X}$ with initial point y_0 such that $pg = f$. Let y_1 denote the terminal point of g. Then, $y_0 \to y_1$ is the desired mapping. Using the inverse path $\bar{f}$ [defined by $\bar{f}(t) = f(1 - t)$], we can define in an analogous way a map $p^{-1}(x_1) \to p^{-1}(x_0)$. It is clear that these maps are the inverse of each other; hence each is one-to-one and onto. Q.E.D.

This common cardinal number of the sets $p^{-1}(x)$, $x \in X$, is called the *number of sheets* of the covering space $(\tilde{X}, p)$. For example, we speak of an n-sheeted covering, or an infinite-sheeted covering.

Examples

3.1. Consider the covering space $(\mathbf{R}, p)$ of S^1 described in Example 2.1. According to Lemmas 3.1 and 3.3, any element $\alpha \in \pi(S^1, (0, 1))$ can be "lifted" to a *unique* path class in $\mathbf{R}$ starting at the point 0. The end point of this path class will be some integral multiple of 2π. Conversely, suppose we have a path class β in $\mathbf{R}$ starting at 0 and ending at some point which is an integral multiple of 2π. The path class $p_*(\beta)$ is an element of $\pi(S^1)$. According to this argument, path classes in $\mathbf{R}$ which end at different integral multiples of 2π must give rise to different elements of $\pi(S^1)$. Thus, $\pi(S^1)$ is an infinite group. This completes the proof of Theorem 5.1 of Chapter II.