one of those that are known; it belongs to those that we simply call linear loci (without knowing anything more about their nature or their properties). Nobody has made the synthesis of these loci, nor has anyone shown how they serve as such loci, not even for the one that would seem to be the first and most obvious. These loci appear in the following proposition.

"If there are six given lines, and the ratio of the solid with three of the drawn lines as sides to the solid with the other three lines as sides is given, then the point will also be on a line given in position.

"If there are more than six lines, then we can no longer say that we give the ratio between some object based on four lines and another object based on the other lines, because there is no figure that can be based on more than three dimensions.[3] And yet certain recent writers have permitted themselves to interpret such things, but when they referred to the product of a rectangle by a square or a rectangle they ceased being intelligible. Yet they might have expressed and indicated their meaning generally by means of compound ratios, both in the case of the previous propositions and in the case of those now under discussion, in the following way.

"Through a point we draw two lines given in position to other lines at given angles, and the product consisting of the ratio of one of those drawn lines to another, and of the ratio of another couple of these drawn lines, and that of a third couple, and finally that of the last drawn line to a (specially) given line—if there are seven lines in all—or of that of the two last ones, if there are eight of them, is given. Then the point will be on a line given in position.

"The same can be said for any number of lines, even or odd. But, as I have said, for each of these loci which follow that for four lines, there has not been made any synthesis which permits us to know the line."

The problem of Pappus can be stated in modern terms as follows [Fig. 1] (it is understood that it is a problem in the plane). If to a line $L(x, y) = ax + by + c = 0$ a line M is drawn through a point $P(x_0, y_0)$ at angle α intersecting the line $L = 0$ in Q, then $PQ = \pm L(x_0, y_0) \times \operatorname{cosec} \alpha / \sqrt{a^2 + b^2} = \text{const.}\ L(x_0, y_0)$. Hence if the $2n$ lines are given by the equations $L_i = 0, M_i = 0, i = 1, 2, \ldots, n$, then the locus of P is given by the equation $L_1L_2 \cdots L_n \pm \lambda M_1M_2 \cdots M_n = 0$, the value of λ depending on the given α, and to a solution with positive λ there always corresponds another one with negative λ. When $2n - 1$ lines are given: $L_i = 0$, $i = 1, \ldots, n$, $M_a = 0$, $a = 1, \ldots, n - 1$, then the locus is given by some

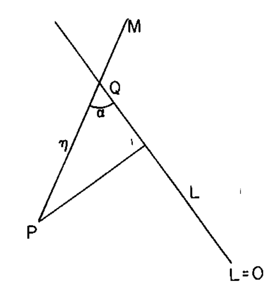


Fig. 1

[3] Acceptance of a geometry of four dimensions had to wait until the nineteenth century. See F. Cajori, *A history of mathematics* (Macmillan, New York, 2d ed., 1938), 184, 256, and Selection III.1, note 2. The first to build a systematic geometry of more than three dimensions was H. Grassmann, in his *Ausdehnungslehre* of 1844.

equation of the form $L_1L_2 \cdots L_n \pm \mu M_1M_2 \cdots M_{k-1}M_k^2M_{k+1} \cdots M_{n-1} = 0$. For $n = 2$ the locus is a conic section.

It should be noted that in the title of Fermat's paper selected here: *Ad locos planos et solidos isagoge* (Introduction to plane and solid loci; *Oeuvres*, I, 92–103, French translation, III, 85–96), the term "plane locus" refers to a locus that can be constructed with the aid of straightedge and compass only, the term "solid locus" to one in which a conic section different from a circle or a straight line appears. When curves of degree higher than two appeared, the problem, or locus, was called linear. These terms appear in Pappus, and not only Fermat but also Descartes and others used them (see Selection III.4). A modern translation of the title would be: *Introduction to loci consisting of straight lines and curves of the second degree.*

There is no doubt that several ancient authors have written on loci, witness Pappus, who, at the beginning of his seventh book, states that Apollonius had written on plane loci and Aristaeus on solid loci.[4] But it seems that to them the study of loci did not come quite easily; this we gather from the fact that for several loci they did not give a sufficiently general account, as will be clarified by what follows here.

We shall therefore submit this science to an appropriate and particular analysis, so that from now on a general way to the study of loci shall be opened.

As soon as in a final equation [*aequalitas*] two unknown quantities appear, there exists a locus, and the end point of one of the two quantities describes a straight or a curved line. The straight line is the only one of its kind, but the types of curves are infinite: a circle, a parabola, a hyperbola, an ellipse, etc.

Whenever the end point of the unknown quantity describes a straight line or a circle, we have a plane locus; when it describes a parabola, hyperbola, or ellipse, then we have a solid locus; if other curves appear, then we say that the locus is a linear locus [*locus linearis*]. We shall not add anything to this last case, since the study of the linear locus can easily be derived from that of plane and solid ones by means of reductions.

The equations can be easily visualized [*institui*], when the two unknown quantities are made to form a given angle, which we usually take to be a right one, with the position and the end point of one of them given. Then, if no one of the unknown quantities exceeds a square, the locus will be plane or solid, as will be clear from what we shall say.

Let NZM be a straight line given in position, N a fixed point [Fig. 2] on it. Let NZ be one unknown quantity A, and the segment ZI, applied to it at given angle NZI, be equal to the other unknown quantity E. When D times A is equal to B times E, the point I will describe a straight line given in position, since B is to D as A is to E.[5] Hence the ratio of A to E is given, and, since the angle at Z is given, the form of the triangle NIZ, and with it the angle INZ, is given. But the point N is given and the straight line NZ is given in position: hence NI is given in position and it is easy to make the synthesis [*compositio*].

[4] Aristaeus flourished about the end of the the fourth century B.C.

[5] *D in A aequetur B in E, ut B ad D, ita A ad E.* Let $A = x = NZ$, $E = y = ZI$; then $Dx = By$, or $B : D = x : y$, which is the equation of line NI.

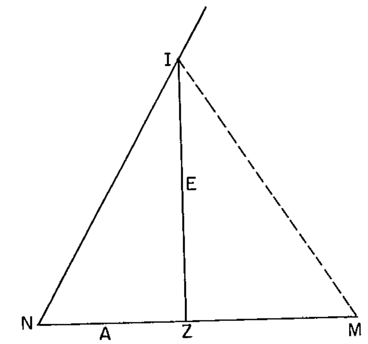


Fig. 2

To this equation all equations can be reduced of which the terms [*homogenea*] are partly given, partly mixed with the unknowns A and E, either multiplied with the given quantities, or appearing simply. Let Z pl $-$ D times A equal B times E. Let D times R be Z pl. Then we will find that B is to D as $R - A$ is to E. Let us take MN equal to R, then point M is given, hence MZ is equal to $R - A$. Hence the ratio of MZ to ZI is known, but the angle at Z is given, hence also the form of the triangle IZM. We conclude that the straight line MI is given in position. Thus point I will be on a straight line given in position.[6] We reach the same result without difficulty for any equation containing the terms A and E.

This is the first and simplest equation of a locus, by means of which all the loci dealing with a straight line can be found; see, for example, the seventh proposition of Book I of Apollonius *On plane loci*, which has since found a more general formulation and construction. This equation also leads to the following elegant proposition, which we discovered with its help:

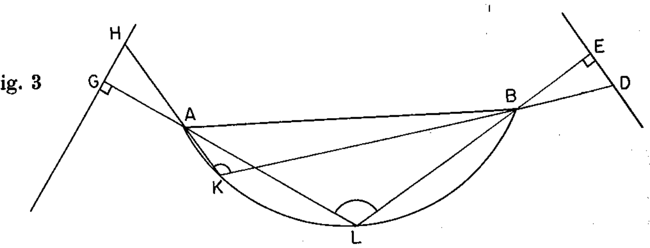
Let any number of straight lines be given in position. From some point draw to them straight lines at given angles. If the sum of the products of the lines thus drawn with the given lines is equal to a given area, then the point will be on a straight line given in position.[7]

[6] When $Z - Dx = By$ (Z, Dx, By are rectangles), then if $Z = DR$ (R a line), $D(R - x) = By$, or $B : D = (R - x) : y$; Z pl, we have seen, means Z is a plane (area).

[7] Fermat was one of the mathematicians who tried to reconstruct Apollonius' book *On plane loci* with the aid of the detailed accounts of it preserved by Pappus. Fermat's reconstruction is in the *Oeuvres*, I, 3–51. Proposition 7 is: "If through two given points at a given angle two lines are led in given ratio, and the endpoint of one line stays on a plane locus [hence a straight line or circle] given in position, then this will be the same for the endpoint of the other."

Hence, if A and B (Fig. 3) are the given points, AH and BD are drawn at the given angle AKB, and AH/BD is in the given ratio, then, if H moves on line HG, D moves on the straight line DE. Here $AG \perp HG$, $BE \perp ED$. This follows from $AH/BD = AG/BE$.

Fig. 3



We omit a great number of other propositions, which could be considered as corollaries to those of Apollonius.

The second species of equations of this kind are of the form

$$A \text{ times } E \text{ is } Z \text{ pl},$$ [8]

in which case point I traces a *hyperbola*. Draw NR parallel to ZI; through any point, such as M, on the line NZ, draw MO parallel to ZI. Construct the rectangle NMO equal in area to Z pl. Through the point O, between the asymptotes NR, NM, describe a hyperbola; its position is determined and it will pass through point I, since we have assumed, as it were, AE—that is to say, the rectangle NZI—equivalent to the rectangle NMO [Fig. 4].

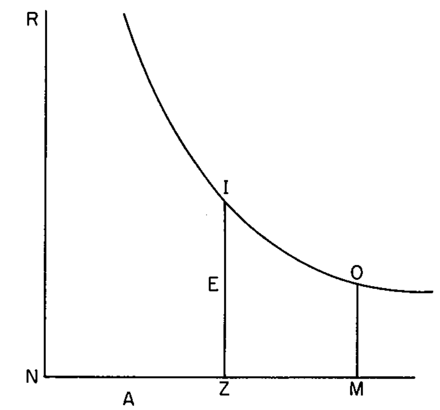


Fig. 4

To this equation we may reduce all those whose terms are in part constant, or in part contain A or E or AE.

If we let

$$D \text{ pl} + A \text{ times } E \text{ equal } R \text{ times } A + S \text{ times } E,$$

then we obtain, by well-known methods,

$$R \text{ times } A + S \text{ times } E - A \text{ times } E \text{ equal } D \text{ pl}.$$

Let us construct a rectangle on two sides such that the terms R times $A + S$ times $E - A$ times E are contained in it; then the two sides will be $A - S$ and $R - E$ and the rectangle on them will be equal to R times $A + S$ times $E - A$ times $E - R$ times E.

If now we subtract R times S from D pl, then the rectangle on $A - S$ and $R - E$ will be equal to D pl $-$ R times S.

Take NO equal to S and ND, parallel to ZI, equal to R. Through point D, draw DP parallel to NZ; through point O, draw OV parallel to ND; prolong ZI to P [Fig. 5].

[8] $xy = Z$.

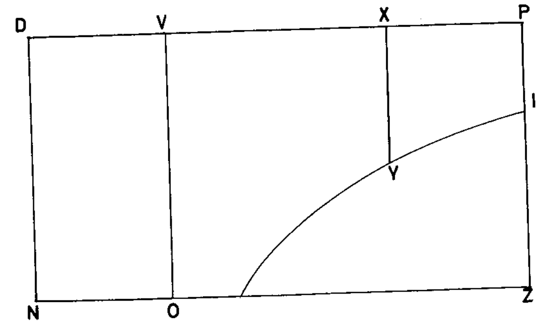


Fig. 5

Since $NO = S$ and $NZ = A$, we have $A - S = OZ = VP$. Similarly, since $ND = ZP = R$ and $ZI = E$, we have $R - E = PI$. The rectangle on PV and PI is therefore equal to the given area D pl $- R$ times E. The point I is therefore on a hyperbola having PV, VO as asymptotes.[9]

If we take any point X, the parallel XY, and construct the rectangle VXY, and through point Y we describe a hyperbola between the asymptotes PV, VO, it will pass through point I. The analysis and construction are easy in every case.

The next species of equations for loci arises if we have A^2 equal to E^2, or in given ratio to E^2, or, again if $A^2 + A$ times E is in given ratio to E^2. Finally this type includes all the equations whose terms are of the second degree containing either A^2, E^2, or the rectangle on A and E. In all these cases the point I traces a *straight line,* as can easily be demonstrated.[10]

Other cases leading to a straight line are $(x^2 + xy) : y^2 = a^2 : b^2$ (Fermat considers only positive values). Then it is shown that the cases which we write in the form $x^2 = ay$, $y^2 = ax$, $b^2 - x^2 = ay$, $b^2 + x^2 = ay$ all lead to parabolas. Then follows $b^2 - x^2 = y^2$, which leads to a circle, as well as $b^2 - 2ax - x^2 = y^2 + 2cy$.

But if $Bq - Aq$ is to Eq in a given ratio, then the point I will be on an *ellipse.*

Let MN be equal to B, and let an ellipse be described with M as vertex, NM as diameter, and N as center, of which the ordinates [*applicatae*] are parallel to the straight line ZI. The squares of the ordinates must have a given ratio to the rectangle formed by the segments of the diameter. Then the point I will be on this ellipse. Indeed, the square on NM − the square on NZ is equal to the rectangle formed by the segments of the diameter [Fig. 6].[11]

9 Let $D + xy = Rx + Sy$, then $Rx + Sy - xy = D$, or $(x - S)(R - y) = D - RS$, reducible to the case of note 6.

10 $x^2 = y^2$, $x^2 : y^2 = a^2 : b^2$, $(x^2 + xy) : y^2 = a^2 : b^2$ all lead to straight lines.

11 This is the case $(b^2 - x^2) : y^2 = p^2 : q^2$. If $MN = b = M'N$, $NZ = x$, then $(b - x)(b + x)/y = \text{const.} = (MZ : M'Z)/IZ$. This way of defining an ellipse is found, for instance, in Archimedes.

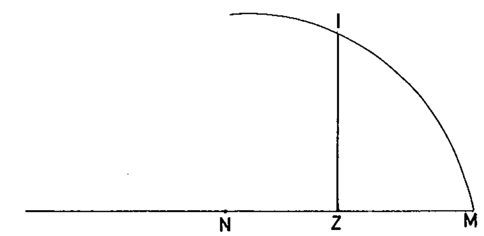


Fig. 6

To this equation can be reduced all those in which Aq is on one side of the equation and Eq with an opposite sign and a different coefficient on the other side. If the coefficients are the same and the angle a right angle, the locus will be a circle, as we have said. If the coefficients are the same, but the angle is not a right angle, the locus will be an ellipse.

Moreover, though the equations include terms which are products of A or E by given magnitudes, the reduction may nevertheless be made by the artifice which we have already employed.

When $(b^2 + x^2) : y^2$ is a given ratio, I lies on a hyperbola. Then follows "the most difficult of all equations," which contains not only x^2 and y^2, but also xy. Fermat analyzes the case $b^2 - 2x^2 = 2xy + y^2$, which, as he shows, represents an ellipse.

Finally Fermat returns to Apollonius' book on plane loci, and at the end solves one more problem on loci:

A single example will suffice to indicate the general method of construction. Given two points N and M, required the locus of the points such that the sum of the squares of IN, IM shall be in a given ratio to the triangle INM [Fig. 7].

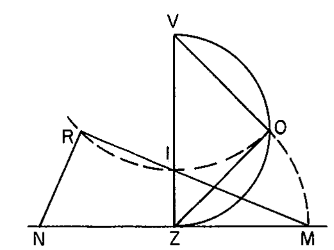


Fig. 7

Let $NM = B$, let E be the line ZI drawn at right angles to NM, and let A be the distance NZ. According to well-known methods we find that A^2 bis $+ B^2 - B$ times A bis $+ E^2$ bis is to rectangle B times E in a given ratio.[12] Following in treatment the procedures previously explained we have the suggested construction.

12 $NM = B$, $ZI = E = y$, $NZ = A = x$. Then $(2x^2 + B^2 - 2Bx + 2y) : By = \text{const.}$

Bisect NM at Z; erect at Z the perpendicular ZV; make the ratio $4ZV$ to NM equal to the given ratio. On VZ draw the semicircle VOZ, inscribe ZO equal to ZM, and draw VO. With V as center and VO as radius draw the circle OIR. If from any point R on this circle, we draw RN, RM, I say that the sum of the squares of RN and RM is in the given ratio to the triangle RNM.

The constructions of the theorems on loci could have been much more elegantly presented if this discovery had preceded our previous revision of the two books *On plane loci*. Yet, we do not regret this work, however precocious or insufficiently ripe it may be. In fact, there is for science a certain fascination in not denying to posterity works that are as yet spiritually incomplete; the labor of the work, at first simple and clumsy, gains strength as well as stature through new inventions. It is quite important that the student should be able to discern clearly the progress which appears veiled as well as the spontaneous development of the science.

---

4 DESCARTES. THE PRINCIPLE OF NONHOMOGENEITY

Descartes, as we have seen (Selections II.7, 8), presented his application of algebra to geometry in the *Géométrie*, published in 1637 as Appendix I to his *Discours de la méthode*. We present here the beginning of Book I of this *Géométrie*, where Descartes explains his principle of nonhomogeneity, based on the proportions $1:a = a:a^2$, $a:a^2 = a^2:a^3$, and so on, which leads him to a notation that is substantially the same as we use in our modern theory of equations, in which we have no hesitation in writing, say, $x^3 + ax^2 + bx + c = 0$ instead of $x^3 + ax^2 + b^2x + c^3 = 0$, and use x, y, z for the unknowns, a, b, c for the given quantities. Descartes then applied his reformed algebra to the geometry of the Ancients, which led to coordinate geometry. Our translation is based on the same Smith-Latham text on which Selection II.8 is based. The title of Book I is *Problems the construction of which requires only straight lines and circles.*

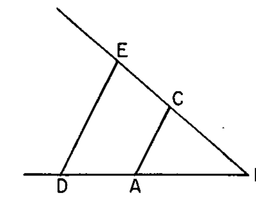
All problems in geometry can easily be reduced to such terms that a knowledge of the lengths of certain straight lines is sufficient for their construction.

Just as arithmetic consists of only four or five operations, namely, addition, subtraction, multiplication, division, and the extraction of roots, which may be considered a kind of division, so in geometry, to find required lines it is merely necessary to add or subtract other lines; or else, taking one line which I shall call the unit in order to relate it as closely as possible to numbers, and which can in general be chosen arbitrarily, and having given two other lines, to find a fourth line which shall be to one of the given lines as the other is to the unit (which is the same as multiplication); or, again, to find a fourth line which is to one of the given lines as the unit is to the other (which is equivalent to division); or, finally, to find one, two, or several mean proportionals between the unit and some other line (which is the same as extracting the square root, cube root, etc.

of the given line). And I shall not fear to introduce these arithmetical terms into geometry, for the sake of greater clarity.

For example, let AB [Fig. 1] be taken as the unit, and let it be required to multiply BD by BC. I have only to join the points A and C, and draw DE parallel to CA; then BE is the product of BD and BC.

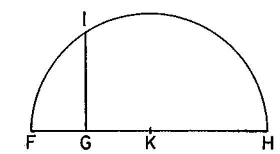
Fig. 1



If it be required to divide BE by BD, I join E and D, and draw AC parallel to DE; then BC is the result of the division.

Or, if the square root of GH [Fig. 2] is desired, I add, along the same straight line, FG equal to the unit; then, bisecting FH at K, I describe the circle FIH about K as a center, and draw from G a perpendicular and extend it to I, and GI is the required root. I do not speak here of cube root, or other roots, since I shall speak more conveniently of them later.

Fig. 2



Often it is not necessary thus to draw the lines on paper, but it is sufficient to designate each by a single letter. Thus, to add the lines BD and GH, I call one a and the other b, and write $a + b$. Then $a - b$ will indicate that b is subtracted from a; ab that a is multiplied by b; a/b[1] that a is divided by b; aa or a^2 that a is multiplied by itself; a^3 that this result is multiplied by a, and so on, indefinitely. Again, if I wish to extract the square root of $a^2 + b^2$, I write $\sqrt{a^2 + b^2}$; if I wish to extract the cube root of $a^3 - b^3 + abb$, I write $\sqrt[3]{a^3 - b^3 + abb}$,[2] and similarly for other roots. Here it must be observed that by a^2, b^3, and similar expressions, I ordinarily mean only simple lines, which, however, I name squares, cubes, etc., so that I make use of the terms employed in algebra.

It should also be noted that all parts of a single line should as a rule be expressed by the same number of dimensions, when the unit is not determined in the problem. Thus, a^3 contains as many dimensions as abb or b^3, these being the component parts of the line which I have called $\sqrt[3]{a^3 - b^3 + abb}$. It is not, however, the same thing when the unit is determined, because it can always be understood, even where there are too many or too few dimensions; thus, if it be

[1] Descartes writes $\frac{a}{b}$.

[2] Descartes writes $\sqrt{C.a^3 - b^3 + abb}$.