Mathematical World • Volume 6

Knots and Surfaces

A Guide to Discovering Mathematics

David W. Farmer
and
Theodore B. Stanford

American Mathematical Society

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Networks

1.1 Countries of the insect world. Imagine a world populated by semi–intelligent insects. The world of the insects is divided into small countries, each country consisting of a few cities connected by dark narrow tunnels. In the course of their work and leisure the insects slowly walk these tunnels, and by the time they reach adulthood all insects know how their country's cities are connected. If an insect needs to travel from one city to another, and those cities are directly connected, then the connecting tunnel is taken. Maybe the route could be shortened by taking two short tunnels through another city, but the insects are only semi–intelligent, so this possibility never occurs to them. And the insects are poor at measuring distances, so they probably couldn't identify a shorter route even it they looked for it. Life on the insect world is calm and uneventful, the citizens blissfully bumping along in the dark, performing their chores with calm inefficiency.

Let's take a closer look at the world of the insects. Here are two insect countries:

Our view from the 'outside' provides us with a complete picture of both countries. The insects are confined to the cities and tunnels, so they must expend more effort to get an accurate view of the layout. Suppose that communication between insect countries takes place by radio. Citizens from the above countries were talking, and they began to wonder if their two countries are the same. How can they determine that their countries have different layouts? First they observe that both countries have four cities and four main tunnels. So far, their countries appear similar. Then one says, "We have a city with just one tunnel leading to it." The other one says, "AHA! All our cities have two tunnels connected to them, so our countries are not set up the same way."

There are many other ways the insects could determine that their countries have different layouts. For example, each of these descriptions applies to exactly one of the countries above:

"My country has a city which connects directly to every other city."

"In my country, you can travel a route of four different tunnels and end up back where you started."

"In my country, you can travel a route of three different tunnels and end up back where you started."

Since the insects are bad at measuring distances, they are not always able to distinguish between layouts which we would see as different.

Task 1.1.1: Explain why the insects cannot distinguish between this country's layout and the first one shown previously.

Task 1.1.2: For each pair of countries, determine whether the insects would view them as the same or different. For those that are different, describe how the insects can tell them apart. Note: for each pair, the number of cities is the same and the number of tunnels is the same. If this were not the case, then the insects could immediately tell that the two countries had a different layout.

continued...

Task 1.1.3: Devise a precise description of what it means for two countries to be 'the same' as far as the insects are concerned.

Task 1.1.4: An insect says, "My country has seven cities and nine tunnels. One city has just one tunnel connected to it, one city has five tunnels connected to it, two cities have three tunnels connected to them, and the other three cities have two connecting tunnels." Draw two different countries which fit that description, and explain how the insects can tell them apart. How many different countries fit that description?

Advice. As you go through this book, you may find it helpful to keep a record of your thoughts and ideas. Set aside a notebook for this purpose. Put all of your work there, not just the final answers. It is important to keep a record of the entire process you went through as you worked on a problem, including work which didn't seem to lead to an answer. Your failed method on one problem could turn out to be the correct method for another problem. Having all your work in one place will help you see what you have done and will make it easy to find old work when you need it.

It is important that you spend sufficient time thinking about the Tasks as you encounter them. Some Tasks are easy and some are very difficult, so you should not expect to find a complete answer to every one. If a Task seems mysterious, it can help to discuss it with someone else. Occasionally you may skip a Task and come back to it later, but skipping a Task in the hope of finding the answers in the text will lead you nowhere. The only way for you to find an answer is to discover it yourself. Sometimes this will mean spending a long time on one Task. That is the nature of mathematical discovery. You will find that discovering your own mathematics is not at all like trying to learn mathematics which has already been discovered by someone else.

1.2 Notation, and a catalog

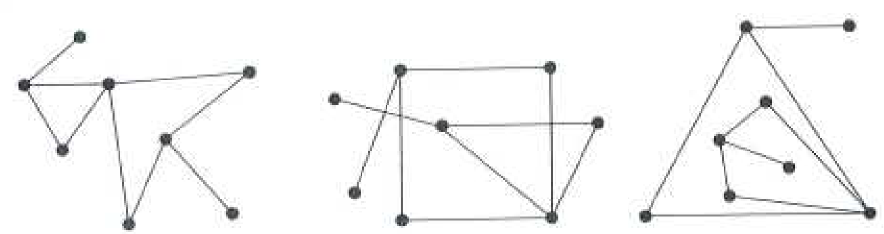
The ideas of the previous section fall under the mathematical topic of *graph theory*. The fanciful idea of insects crawling through dark tunnels will continue to be useful, but we will switch to using the mathematical terminology. Here is how to translate:

Insect name:	Math name:
country	graph
city	point or vertex
tunnel	line or edge

An example sentence is, "A graph is made up of points and lines." Note that 'vertices' is the plural of 'vertex,' so we can also say, "A graph consists of vertices connected by edges."

The actual picture we draw of a graph is called a **graph diagram**. Just as the insects could not distinguish between certain countries, the same graph can be represented by many different graph diagrams. The only important feature of the graph is how the various vertices are connected. Each graph diagram will have additional features, such as the lengths of the edges and the relative position of the vertices, but these aspects of the diagram have nothing to do with the graph itself.

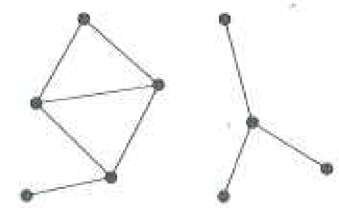
Here are three diagrams of the same graph:



A diagram may appear to show two edges crossing, but if there is not a vertex at the junction then the edges do not actually meet. Think of it as two insect tunnels which pass each other but do not intersect. The topic of drawing graphs without crossing edges will be explored in a later section.

A graph is called **connected** if we can get from any vertex to any other vertex by traveling along edges of the graph. The opposite of connected is **disconnected**.

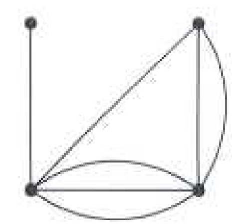
This can be thought of as a disconnected graph with 9 vertices, or as two separate connected graphs.



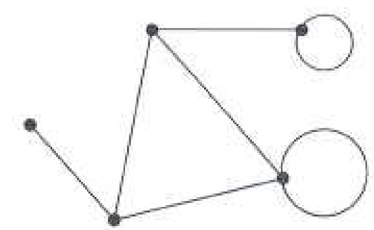
Any graph is just a collection of connected graphs; these are called the **components** of the graph.

The graphs we have been studying are presented as drawings on paper. It is easy to invent graphs which are described in other ways. For example, we can make a graph whose vertices are all of the tennis players in the world, and where an edge connects two players if they have played tennis together. We have defined a graph, although it would not be feasible to actually draw it. Another graph can be made by letting the vertices be the countries of the world, and having an edge connect two countries if those countries border each other. With the help of a map it would be possible to draw this graph. It is amusing to invent fanciful graphs and then try to determine what properties they have. For example, is the tennis player graph connected? If it is, that would mean each tennis player has played someone who has played someone who has . . . played Jimmy Connors. The play *Six Degrees of Separation* mentions, informally, the graph whose vertices are all of the people in the world, with edges connecting people who know each other. The title of the play comes from speculation that you can get from any one vertex to any other vertex by crossing at most 6 edges.

In order to make an organized study of graphs, we must impart a few more rules. Usually we do not allow our graphs to have more than one edge connecting two vertices, and we do not allow an edge to connect a vertex to itself.



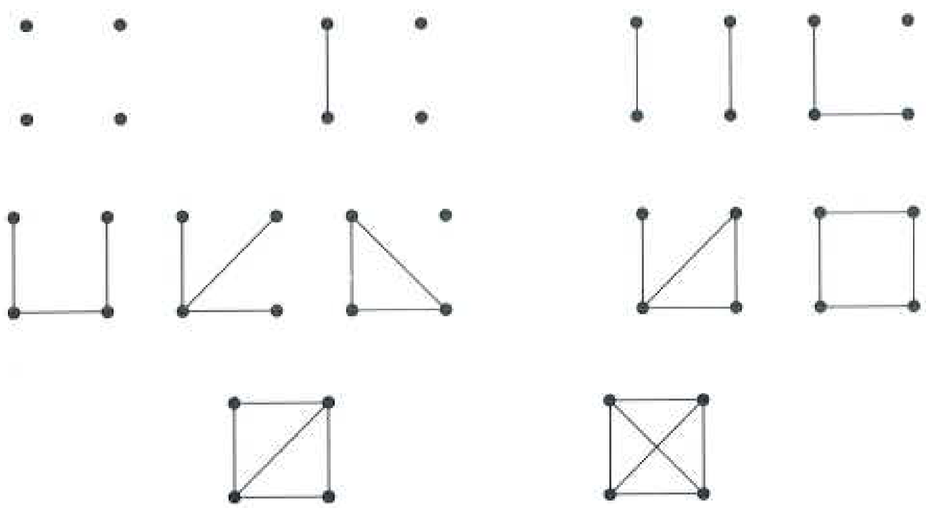
A graph with multiple edges



A graph with loops

Unless we state otherwise, a 'graph' is a 'graph without loops or multiple edges.'

We classify graphs according to how many vertices they have. Here is a catalog of all graphs with 4 vertices:



You should convince yourself that the list is complete.

Task 1.2.1: Make a catalog of all graphs with 5 vertices. Hint: there are between 30 and 40 of them. First find all the ones with no edges, then 1 edge, then 2 edges, and so on.

In the above Task it is difficult to be absolutely sure that you found all the graphs. Fortunately, there is something we can do to increase our confidence. For the graphs with 4 vertices we found a total of $1+1+2+3+2+1+1=11$ graphs, where we counted the graphs according to how many edges they have. Notice that the numbers form a symmetric pattern.

Task 1.2.2: Do your numbers from Task 1.2.1 form a symmetric pattern? If not, go back and fix your list. After your list is correct, explain why the symmetric pattern appears.

Task 1.2.3: Devise a code for describing a graph over the telephone. Note: your code only needs to describe a *graph*, not a *graph diagram*.

1.3 Trees

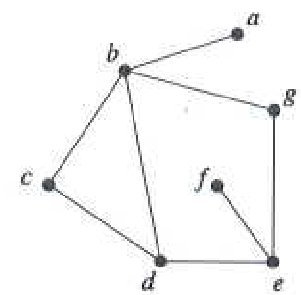
If we think of a graph as a roadmap then it is natural to look at the various routes we can take through the graph. A **path** in a graph is a sequence of edges, where successive edges share a vertex. To make things easier to read, we will describe a path by showing which vertices the path visits; this should not cause any confusion.

Example paths:

(d, b, c, d)

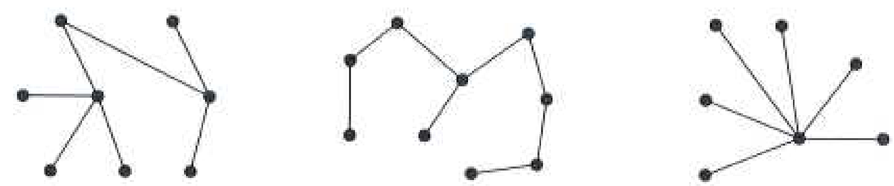
(f, e, d, b, c)

(e, g, b, a, b, g)



A path is **closed** if it ends at the same vertex as it began. The first path above is closed. A path is **simple** if it doesn't use the same edge more than once. The first two paths above are simple. A simple closed path is sometimes called a **circuit**. The first path above is a circuit, and so is *(b, c, d, e, g, b)*. A graph is **connected** if there is a path from any one vertex to any other vertex.

A graph is a **tree** if it is connected and it doesn't have any circuits. Here are three trees:



Task 1.3.1: What is the relationship between the number of vertices and the number of edges in a tree? Why does this relationship hold?

Trees are particularly simple kinds of graphs, so our plan is to study trees, and then to use trees to study other graphs. Here are all trees with 5 vertices:

Those trees should be in your catalog of graphs from Task 1.2.1. Here are all trees with 6 vertices:

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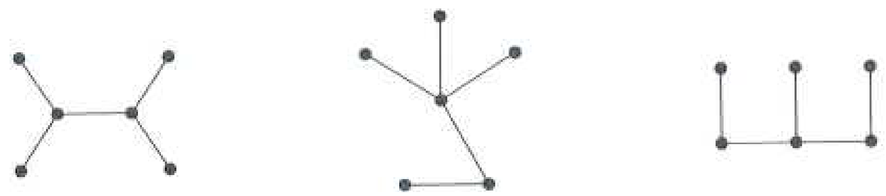


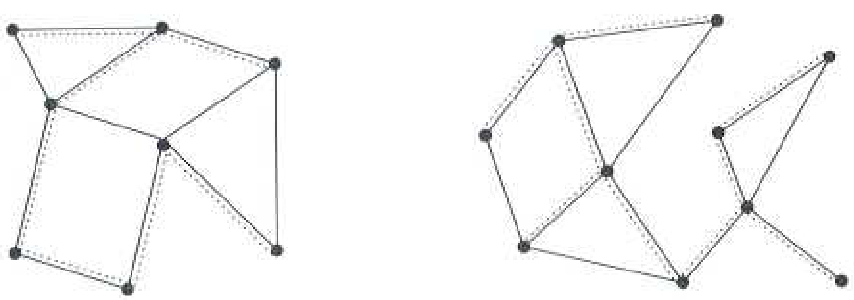
Task 1.3.2: Make a list of all trees with 7 vertices. If you feel ambitious, make a list of all trees with 8 vertices. Hint: there are between 20 and 30 of them.

Task 1.3.3: Suppose you had plenty of time and you wanted to make a list of all trees with a given large number of vertices; say, all trees with 12 vertices. Describe the method you would use. Is your method guaranteed to give a complete list with no repeats? Is your method practical?

Task 1.3.4: In Task 1.2.3 you devised a code for describing a graph over the telephone. Suppose that you only needed the code for describing trees. Is it possible to devise a simpler code which still works in this case?

1.4 Trees in graphs

A tree inside a graph which hits every vertex of the graph is called a **spanning tree**. A spanning tree must use the edges in the graph, and it must hit every vertex. A useful way to show a spanning tree is to highlight the edges in the tree:



A graph can have many different spanning trees. Here are three different spanning trees for the same graph:

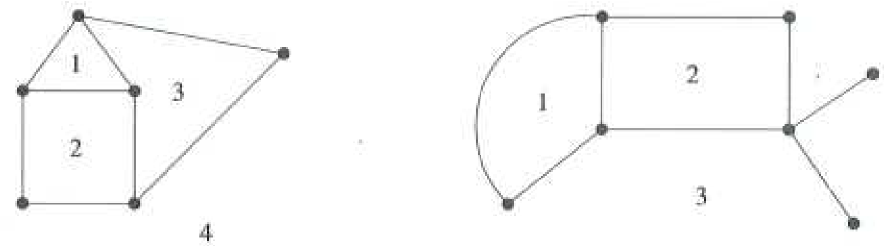
It is important to keep in mind that a graph can have several different spanning trees, so without a picture the term 'spanning tree' can be ambiguous.

Task 1.4.1: Devise a way of counting the number of spanning trees of a graph.

In the next section we use spanning trees to study graphs.

1.5 Euler's formula

A graph diagram divides the plane into separate regions:

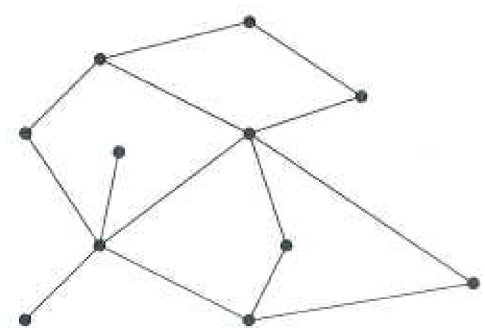


The first diagram divides the plane into 4 regions, and the second divides the plane into 3 regions. Note that the big outside area counts as a region.

Task 1.5.1: Draw several graphs and record the following information:

- the number of vertices in the graph (call it v)
- the number of edges in the graph (call it e)
- the number of separate regions of the graph (call it f)
- the number of vertices in a spanning tree (call it A)
- the number of edges in a spanning tree (call it B)
- the number of edges not in a spanning tree (call it C)

Here is an example. Find a spanning tree and check that the numbers are correct:



$v = 11$
$e = 14$
$f = 5$
$A = 11$
$B = 10$
$C = 4$

Note: for this Task you should only use connected graphs which you have drawn without crossing edges. A diagram drawn without crossing edges is called a

planar diagram. The importance of using planar diagrams in this Task is discussed in the next section.

Task 1.5.2: Look at the information you recorded and try to find patterns and relationships among the six quantities.

Task 1.5.3: Explain why the observations you made are correct. Note: one of your observations may have been $A = B + 1$. You already discussed this in Task 1.3.1.

Task 1.5.4: Explain why your observations can be used to show $v - e + f = 2$.

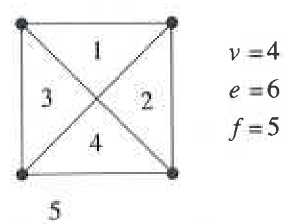
The equation $v - e + f = 2$ is known as **Euler's Formula**. It was first discovered by the Swiss mathematician Leonhard Euler in 1736. Note: Euler is pronounced 'Oiler.' Say it out loud a few times. This will keep you from looking foolish later.

Task 1.5.5: Suppose a graph has 7 vertices and 9 edges. Use Euler's formula to predict how many separate regions it would have if you drew the graph. Draw such a graph and check if your prediction is correct.

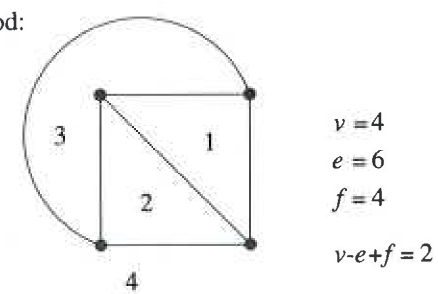
1.6 Planar graphs

Euler's formula $v - e + f = 2$ is true for any connected graph which is drawn without crossing edges. For instance:

Bad:



Good:

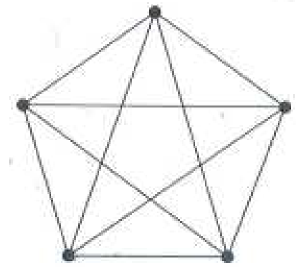


We say that a graph is **planar** if it has a diagram without crossing edges. Above are two diagrams of the same graph, but Euler's formula only works in the second case. This is usually expressed as "Euler's formula holds for connected planar graph diagrams."

Task 1.6.1: What can you say about $v - e + f$ if the graph is not connected?

The graph shown above is called 'the complete graph on 4 vertices,' and it is denoted K_4, pronounced "kay four." This means that it has 4 vertices and each vertex is connected to every other vertex. Similarly, K_5 is the graph with 5 vertices and each vertex is connected to every other vertex.

Here is a representation of K_5:



Task 1.6.2: How many edges does K_5 have? K_6? K_7?

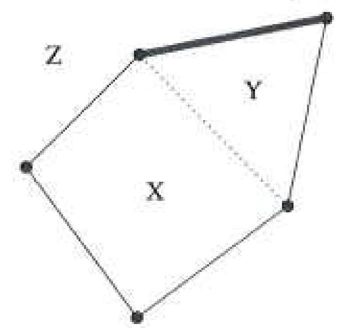
Task 1.6.3: Explain why K_n has $1 + 2 + 3 + \cdots + (n - 1)$ edges. We will find another expression for this in Task 1.7.9.

Task 1.6.4: Try to draw K_5 without any crossing edges. Make at least four attempts.

After four attempts at Task 1.6.4, you should stop. Further attempts would be pointless because it is *impossible* to draw K_5 without any crossing edges. In other words, K_5 is not planar. We will use Euler's formula to show why the Task is impossible.

The reason we write f for the number of separate regions of a graph is that those regions are usually called **faces**. A key fact we need is that each edge of a planar graph diagram is a border of two faces.

The dotted edge is a border of face X and face Y, and the fuzzy edge is a border of faces Y and Z.



The number of edges of a face is called the **order** of the face. In the diagram above, face X has order 4, face Y has order 3, and face Z has order 5.

An important relationship between the number of edges and the number of faces in a planar graph is:

$$3f \leq 2e.$$

We will use the concept of order, along with the observation that each edge borders two faces, to establish this inequality. Then we will use the inequality to show that K_5 does not have a planar diagram. But first, do this Task:

Task 1.6.5: Draw a few planar graph diagrams and check that $3f \leq 2e$ holds in each case. What graphs have $3f = 2e$?

The first step in showing $3f \le 2e$ is this observation:

Counting Observation. If you add up the number of edges bordering every face, then you get twice the number of edges. In other words:

$$\text{the sum of all the orders of the faces} = 2e.$$

For example, in the diagram above there are three faces: X, Y, and Z. Adding up the orders of each faces gives $4+3+5=12$. And sure enough, 12 is twice the number of edges in the graph.

The reason behind the Counting Observation is that each edge is a border of two faces, so as we add up the number of edges around each face, each edge gets counted twice.

Now, we need at least 3 edges to make a face, so each face must have order at least 3. So, adding up the orders of each face gives something at least $3f$, so

$$3f \le \text{the sum of all the orders of the faces},$$

so,

$$3f \le 2e.$$

This is the inequality we wanted. We now use it to show K_5 is not planar.

The graph K_5 has 5 vertices and 10 edges. **IF** K_5 had a planar diagram, then Euler's formula $v-e+f=2$ would tell us $f=7$. The inequality we proved would then say

$$3f \le 2e$$
$$3 \cdot 7 \le 2 \cdot 10$$
$$21 \le 20.$$

Of course, 21 is more than 20, so something is wrong. The error was the assumption that K_5 had a planar diagram. The inescapable conclusion is that K_5 does not have a planar diagram. In other words, it is impossible to draw K_5 with no crossing edges. In other other words, K_5 is not planar.

Task 1.6.6: For each case below, either draw a planar graph with the given information, or explain why this is not possible.

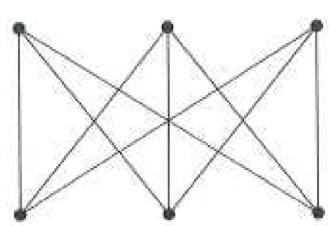
a) A graph with $v=7$ and $e=17$.

b) A graph with $v=8$ and $e=12$.

c) A graph with $v=7$ and $e=15$.

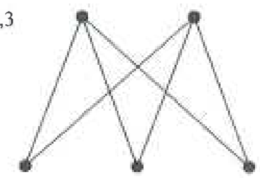
The method we used to show that K_5 does not have a planar diagram can also be used to show that some other graphs are not planar, but the method is not foolproof.

This graph is called $K_{3,3}$. It has 6 vertices and 9 edges, and it does not have a planar diagram.

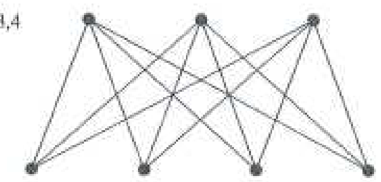


The name $K_{3,3}$, pronounced "kay three three," means the graph consists of one set of 3 vertices, each of which is connected to another set of 3 vertices. Here are more examples to explain the notation:

$K_{2,3}$



$K_{3,4}$



Task 1.6.7: How many edges does $K_{2,5}$ have? $K_{6,6}$? $K_{100,200}$? $K_{n,m}$?

Task 1.6.8: Try to draw $K_{3,3}$ without crossing edges. You will not succeed, because $K_{3,3}$ is not planar, but you should make a few attempts anyway.

We showed that K_5 does not have a planar diagram, but the same method doesn't work for $K_{3,3}$. Fortunately, we can modify the method. The key to the modification is this observation:

No Triangles Observation. $K_{3,3}$ does not contain any simple closed paths of three edges.

The quick way to say it is: $K_{3,3}$ does not contain any triangles. This means that if we could draw $K_{3,3}$ without any crossing edges, then every face would have order at least four.

Task 1.6.9: Explain why a planar graph with no triangles must have $4f \le 2e$. Use this to show that $K_{3,3}$ does not have a planar diagram.

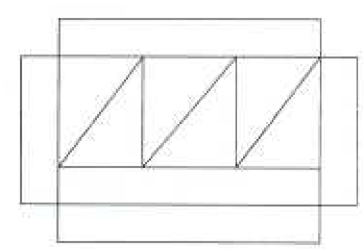
It turns out that understanding K_5 and $K_{3,3}$ is fundamental to understanding all nonplanar graphs. See the Notes at the end of the chapter for an explanation.

Task 1.6.10: Do all trees have a planar diagram?

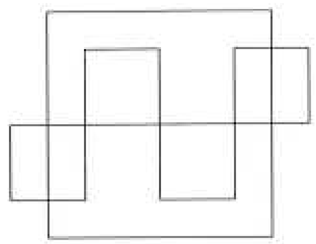
1.7 Paths in graphs

In this section we study various kinds of paths in a graph. We start with a puzzle and two problems.

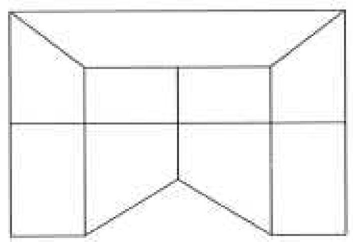
Task 1.7.1: Trace this figure without picking up your pencil and without repeating a line, or explain why this is impossible.



Task 1.7.2: A huge snowstorm has covered the town with snow! There is only one snowplow, and the roads need to be plowed as soon as possible. The snowplow driver figures that if she can manage to plow all the streets in one trip, without driving over a street which has previously been plowed, then this will get the job done quickly. Trace an efficient route on the map shown to the right, or explain why a completely efficient route is impossible.



Task 1.7.3: The highway inspector must evaluate the safety of all the roads in town. Since he is lazy, he wants to travel along each road exactly once; he does not want to drive on a road which he has already inspected. Trace an efficient route for the lazy inspector, or explain why no such route is possible.



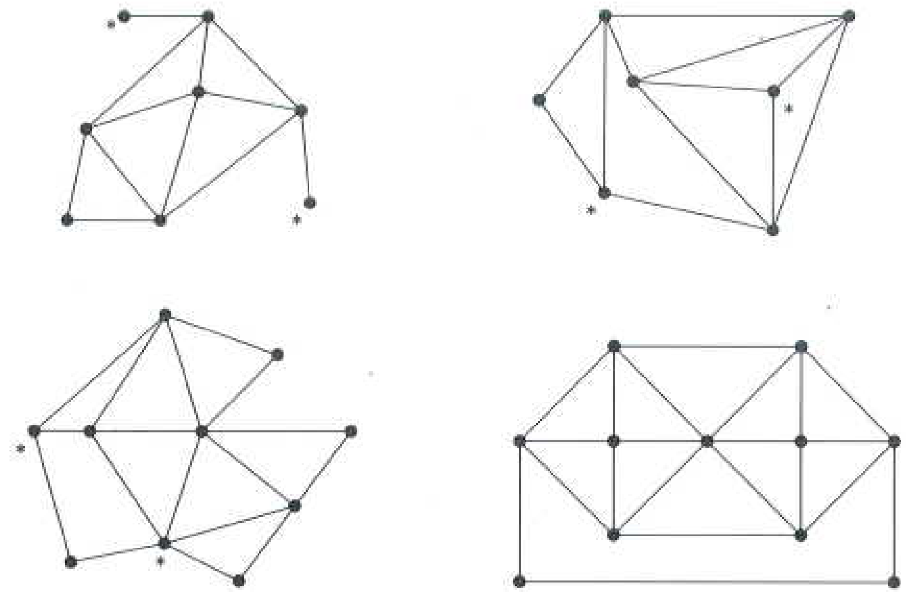
Hopefully you traced the figure in Task 1.7.1 and you also found an efficient path for the snowplow in Task 1.7.2. If not, then go back and try again! There is no efficient path for the lazy inspector in Task 1.7.3. No matter how hard you try, it is impossible to trace the roads of that town without using some road more than once. Our next goal is to give an explanation for this.

The three Tasks above have the same theme: each gives a graph and asks if there is a simple path which uses every edge in the graph. Recall that *simple* means that no edge is used twice. In a graph, a simple path which uses every edge is called an **Euler path**. An **Euler circuit** is an Euler path which begins and ends at the same vertex.

If you check back at your answers to the above Tasks, your solution to the puzzle is an Euler *path*, and your solution to the snowplow problem is an Euler *circuit*. The graph for the highway inspector has neither an Euler path nor an Euler circuit.

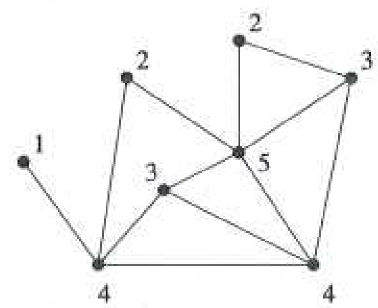
Task 1.7.4: Find an Euler path in each of these graphs. The Task will become

very easy once you determine the significance of the $*$ vertices.



The key to Euler paths in a graph lies in careful examination of the vertices. The number of edges connected to a vertex is called the **order** of the vertex.

Each vertex in this graph is labeled with its order.



Task 1.7.5: What relationship exists between an Euler path in a graph and the orders of the vertices in that graph? If you don't see a relationship yet, then do more examples. Explain why your observations are correct.

The previous Task is the key point of this section. Be sure to give it sufficient thought.

Task 1.7.6: Repeat your work in the previous Task for Euler *circuits*.

Task 1.7.7: Draw a graph with exactly *one* odd-order vertex. Does it have an Euler path?

Task 1.7.7 is a trick question: it is impossible to draw a graph with exactly one odd–order vertex. Here is one way to see this:

Task 1.7.8: Explain why the Counting Observation following Task 1.6.5 is valid with 'face' replaced by 'vertex.' Use this New Counting Observation to explain why Task 1.7.7 is impossible.

Task 1.7.9: Use the New Counting Observation in Task 1.7.8 to show that K_n has $n(n-1)/2$ edges. Compare this to Task 1.6.3.

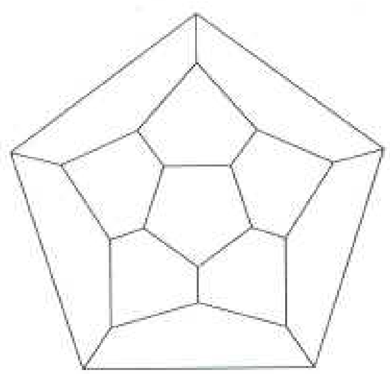
The *Handshake Principle* is a slight generalization of our rule that a graph cannot have exactly one odd–order vertex.

The Handshake Principle. Take a group of people and have each person shake hands with various other people in the group. The number of people who shook hands an odd number of times must be even.

In terms of a graph, the Handshake Principle says that the number of odd–order vertices must be even.

We have been studying paths which cross every edge once. Another interesting problem is to study paths which visit every vertex once. This idea first appeared in a game invented by the English mathematician Sir William Rowan Hamilton. He sold the idea to a game producer, but the game never made any money!

Hamilton's problem: find a path in this graph which visits every vertex once, and which ends at the same vertex as it began. In the original version each vertex was labeled with a famous city, and the object was to 'Travel the World.'



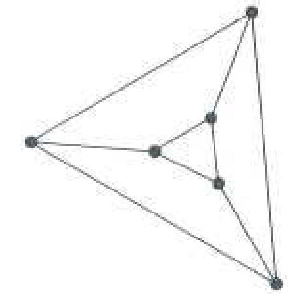
A **Hamiltonian path** in a graph visits every vertex exactly once. The path is a **Hamiltonian circuit** if it ends at the same vertex as it began. A Hamiltonian path can also be thought of as a spanning tree with no 'branches.' See the middle graph at the very end of Section 1.4 for an example.

The study of Hamiltonian paths is much more difficult than the study of Euler paths. A few rules are known, and any specific graph can be analyzed by computer, but nobody has found a simple method for determining when a graph has a Hamiltonian path.

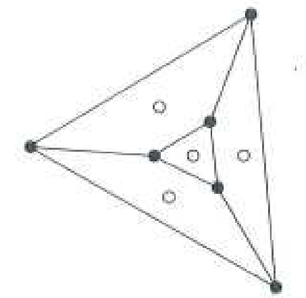
1.8 Dual graphs

We introduce a way to take one graph diagram and use it to produce another graph. This new graph is called the *dual* of the original diagram.

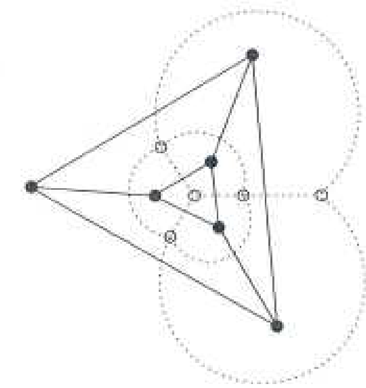
Start with any planar diagram:



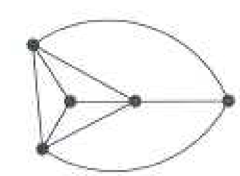
Put a vertex in each separate region of the original diagram.



Connect the new vertices which are in adjacent regions of the original diagram. Each original edge will have a new edge crossing it.



The graph made with the new edges and vertices is called the **dual** of the original graph diagram.



The above procedure is called **taking the dual** of the original graph diagram. We will see that there are interesting relationships between a graph and its dual, and the dual graph is useful for solving certain problems.

Recall that a *graph* is not permitted to have loops or multiple edges. Because of this restriction, the dual of a graph diagram might not be a 'graph' in the strictest sense. For example:

A diagram:

Its dual:

That dual graph has multiple edges, so it is not 'really' a graph. It is too much effort to keep worrying about this distinction, so for the rest of this section we will permit our graphs to have loops and multiple edges.

Task 1.8.1: Draw a few planar graph diagrams, and then find their duals.

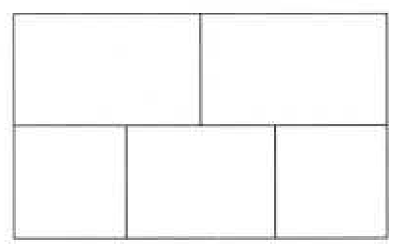
Task 1.8.2: What is the relationship between v, e, f for a graph and v, e, f for its dual?

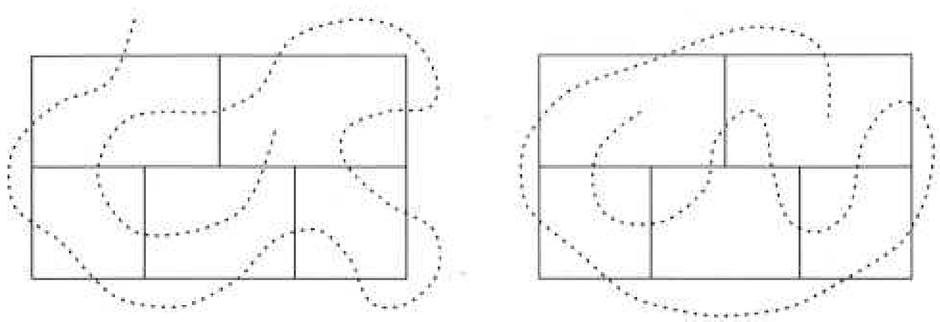
After doing Task 1.8.2, you should look back at the first part of Task 1.7.8.

Task 1.8.3: What is the relationship between a graph, its dual, and the dual of the dual?

Task 1.8.4: Find a few graphs which are the same as their dual.

Task 1.8.5: Can you draw a curve which crosses each edge of this figure exactly once? The curve is not allowed to cross itself. Two failed attempts are shown below.



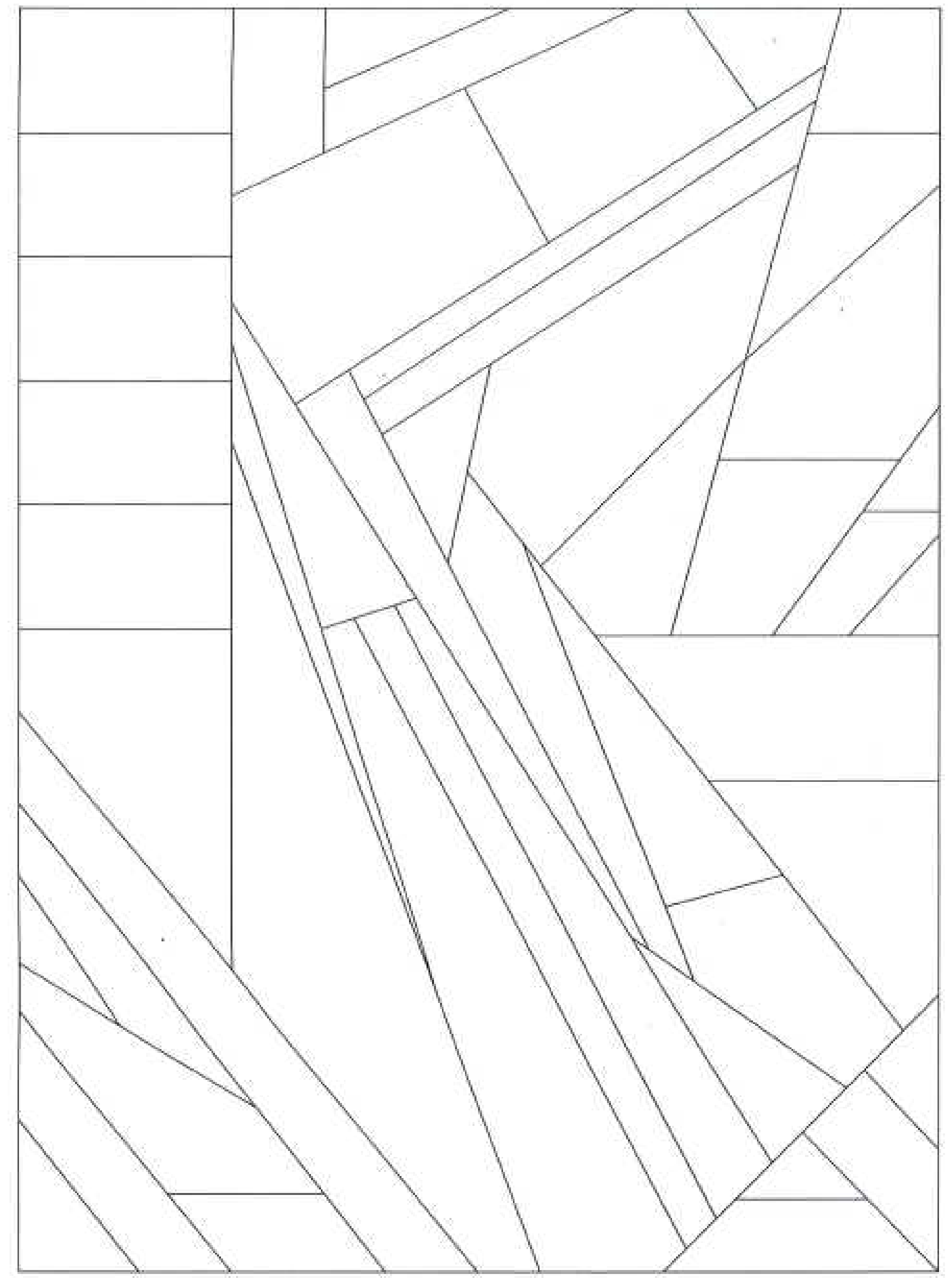


Task 1.8.6: Would the above Task be any easier if the curve was allowed to cross itself?

1.9 A map of the United States

On the next page is a map of the 48 continental US states. It might not

look like a typical US map, but there is a way to label each region with the name of a state so that each state borders the same states as on the usual map.



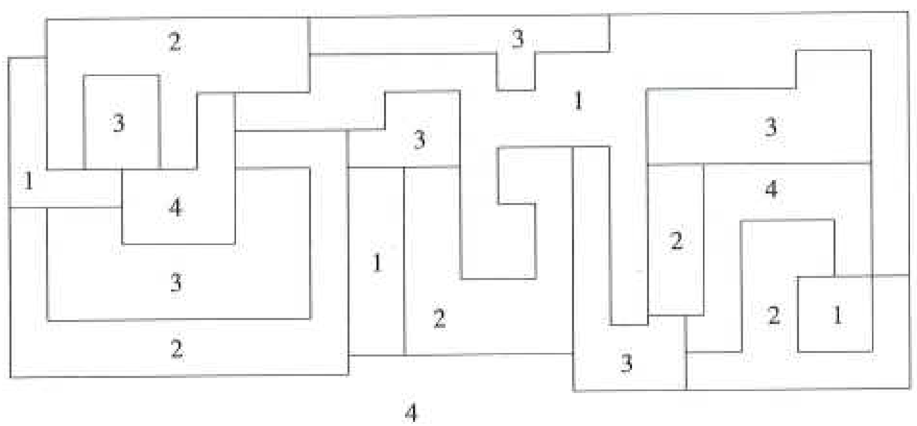
Task 1.9.1: Label the 48 regions of the map. Explain how you determined which region corresponds to which state. Is there only one way to correctly label the regions?

Task 1.9.2: Which state borders the most other states?

Task 1.9.3: Devise a similar map for your favorite geographic region.

1.10 Coloring graphs

A cartographer is producing a new map of the world. To make the map easier to read, each country will be given a color, and adjacent countries must be assigned different colors. Two countries can have the same color provided they don't border each other. Here is a map which we have colored using the numbers 1, 2, 3 and 4:



Note that the large outside region also gets a color.

There are a few rules about the 'maps' we will consider. Each region represents a separate country, and countries are not permitted to have 'satellite states.' For example, on the usual world map we would not permit Alaska to be part of the Unites States, because Alaska is not directly connected to the other states. We can use the world map as an example for our 'map,' but must permit Alaska to be a different color than the 48 continental states. Also, we do not consider two regions to be bordering if they have only one point in common. For example, Colorado and Arizona do not border each other on the United States map. Another example can be seen in the lower right portion of the example above. The small square labeled '1' does not border the large region labeled '1.'

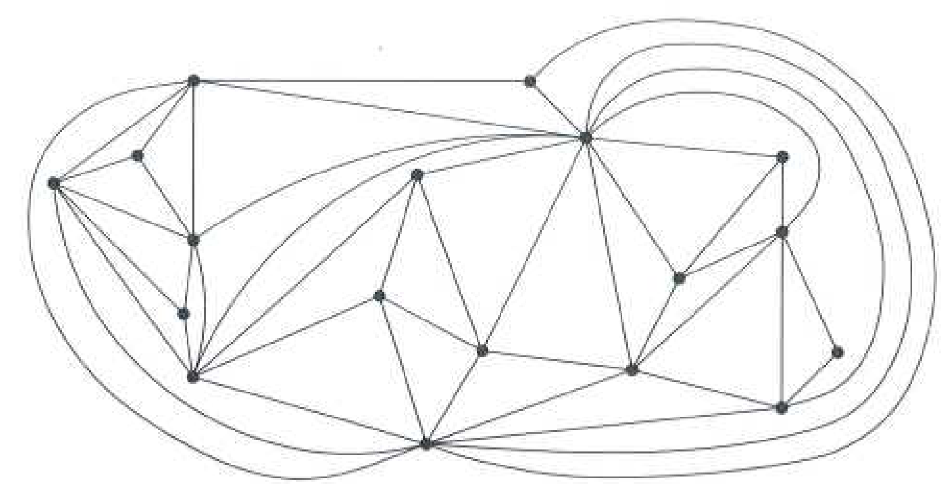
Task 1.10.1: Draw a few maps and color them. Use as few colors as possible.

All of your maps from Task 1.10.1 can be colored with at most 4 colors. If you used 5 or more colors in some of your maps then go back and color them again. All maps can be colored with at most 4 colors. If you are skeptical about this then try to design a map that requires 5 colors.

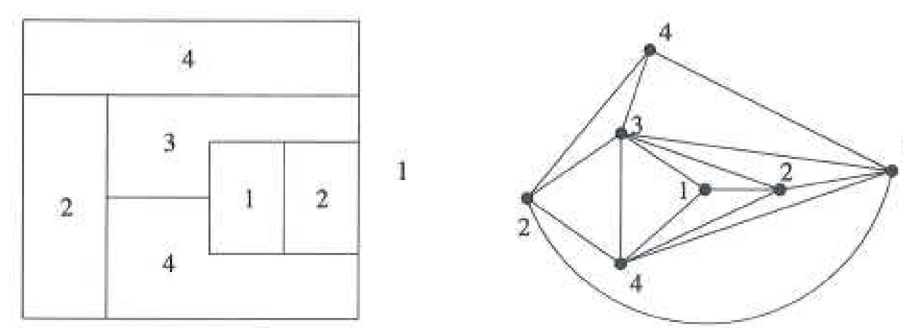
The statement that any map can be 4–colored is known as "The Four–Color Theorem." The Four–Color Theorem is easy to understand, and trying a few examples makes it easy to believe, but giving a complete proof is extremely difficult. The Four–Color Theorem has been widely believed for more than 100

years, but a formal proof was not given until 1974. The proof was controversial because a computer was used for part of the calculation. Some people are still skeptical that all of the details were checked properly, and an independent calculation has not yet been completed. In this book we give a detailed proof that any graph can be 6–colored, and we roughly indicate why any graph can be 5–colored. If you figure out a simple proof of The Four–Color Theorem, then your name will be immortalized forever in the annals of mathematics.

We will spend the rest of this section using graphs to study map colorings. It is easy to associate a graph to a map: put a vertex in each region, including the large outside region, and connect adjacent regions with an edge. This works just like finding a dual graph. Here is the graph associated to the map shown previously:



And here is a simpler example. Each vertex in the graph is colored the same as the corresponding region of the map.

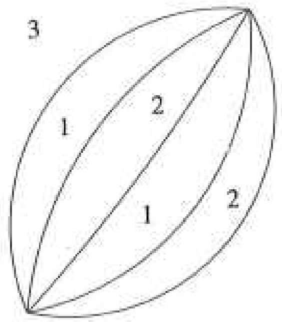


Coloring a map is now replaced by coloring a planar graph: we color the *vertices* of the graph so that adjacent vertices, vertices which share an edge, have different colors. For the purposes of coloring, the graph has the same information as the map. We will study graph coloring because then we can make use of our knowledge of planar graphs.

Task 1.10.2: An obvious way to try to get a map which cannot be 4–colored is to draw 5 countries, each one of which borders the other 4 countries. Since each country borders every other country, 5 different colors would be needed. However, such an arrangement is impossible. Explain why.

It might seem that since we can't have 5 countries simultaneously bordering each other, that would automatically imply that any map can be 4–colored. However, the logic is flawed. To see the flaw, look at the map below.

Each region borders only two other regions, but the map requires three colors.



The above example shows that the number of neighbors of each region is not always directly related to the number of colors needed to color the map. Another version of the same idea is given in the next Task.

Task 1.10.3: Color the vertices of each of the following graphs with as few colors as possible. Note that the number of colors is not directly related to the number of neighbors of each vertex.

For certain kinds of graphs we can say exactly how many colors are needed to color them.

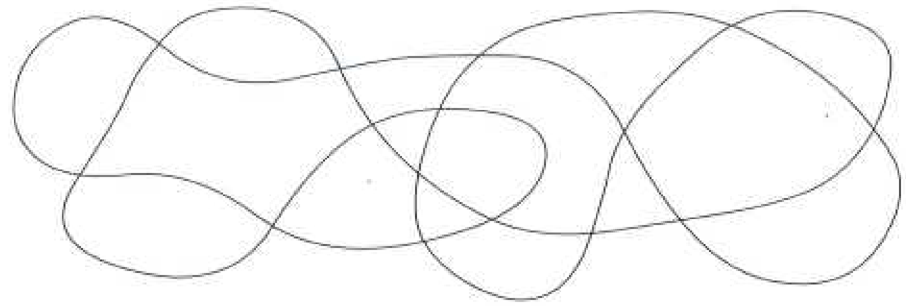
Task 1.10.4: Suppose a graph has every vertex of order 3 or less. Explain how to 4–color that graph.

Task 1.10.5: Explain why any tree can be 2–colored.

Task 1.10.6: Suppose each separate region of a graph has an even number of sides. Is it necessarily true that the graph can be 2–colored? Note: you are

coloring the *graph*, so each vertex gets a color, and vertices sharing an edge must get different colors.

Task 1.10.7: Construct a map by drawing a continuous curve which begins and ends at the same point and crosses itself as many times as you want. An example is shown below. Determine how many colors are needed to color such a map.



Task 1.10.8: Repeat Task 1.10.7 for maps made by several different overlapping curves. That is, draw several different curves as in Task 1.10.7 and permit the curves to cross each other.

Task 1.10.9: How many colors are needed to color K_n? Note: for $n \geq 5$ that graph will not be planar, but it still makes sense to color the vertices so that vertices sharing an edge get different colors.

Task 1.10.10: How many colors are needed to color $K_{n,m}$?

Task 1.10.11: Devise some graphs which require exactly three colors.

1.11 The six–color theorem

Since The Four–Color Theorem is so hard, we will prove The Six–Color Theorem: every graph can be colored with at most six colors. This is a common occurrence in mathematics: the ultimate goal may be out of reach, but we can still get satisfaction by proving a partial result. To prove The Six–Color Theorem, we need this fact about planar graphs.

Planar Graph Fact. Every planar graph has at least one vertex of order five or less.

In other words, we can't have every vertex of order six or more. We will prove The Six–Color Theorem, and after that we will prove the Planar Graph Fact. But first...

Task 1.11.1: Draw a graph where each vertex has order at least five. What does this say about the Planar Graph Fact?

We will describe a procedure for 6–coloring a planar graph, and then we will illustrate the procedure with an example.

To 6–color any planar graph, just follow these four steps:

Step 1: Locate a vertex of order 5 or less.

Step 2: Delete that vertex and all edges connected to it.

Keep repeating Steps 1 and 2 until only 5 vertices are left. Keep track of the order in which you deleted the vertices.

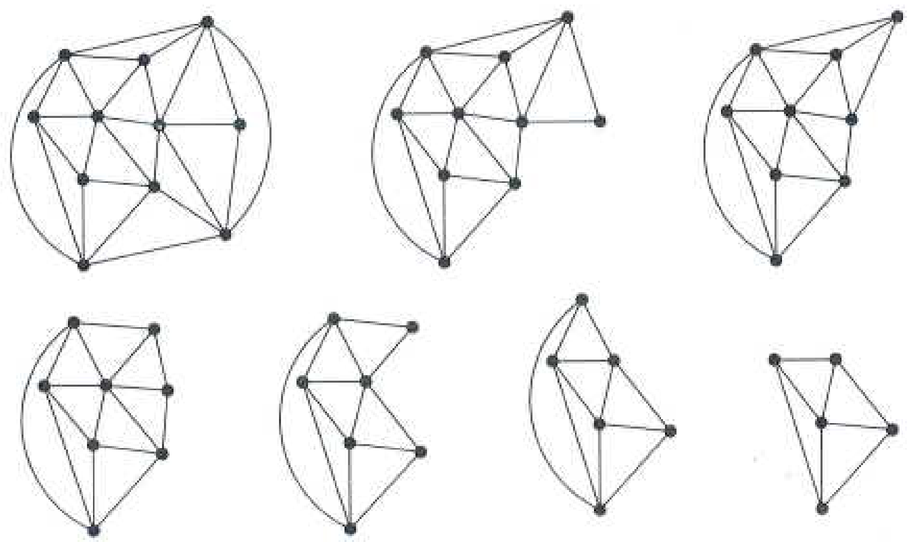
Step 3: Color the five remaining vertices with the colors 1, 2, 3, 4, 5.

Step 4: Put back the last vertex and edges you deleted. Color that vertex a different color than the vertices adjacent to it.

Repeat Step 4, replacing vertices in the reverse order they were deleted. After you put all the vertices back you will have reconstructed the original graph and each vertex will be colored with one of the 6 colors.

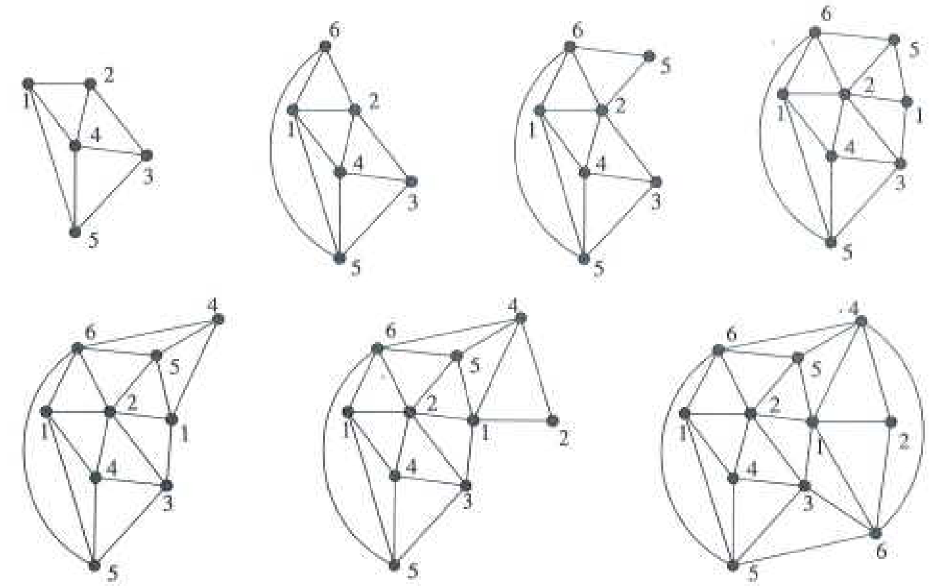
The Planar Graph Fact mentioned above is what makes the procedure work. Since a planar graph must have a vertex of order 5 or less, Step 1 can always be done. As you put the vertices back, each vertex is connected to at most 5 other vertices. Since there are 6 colors available, Step 4 can always be done.

Here is an example of using the procedure to 6–color a graph. The important thing to notice is that the procedure described above is followed exactly. No cleverness is needed. We just mechanically follow the plan and everything will work out right. First remove vertices one–by–one, also removing the connecting edges. If there is more than one vertex of order less than 6 then it doesn't matter which one we choose. Stop when there are only 5 vertices left.



Now color the remaining vertices, and then replace the vertices in the reverse order they were deleted. When we replace a vertex we color it with any number

different from the vertices it is adjacent to.



The result is a 6–coloring of the graph.

Of course, it is also possible to 4–color that same graph, but this requires cleverness. Nobody has ever found a simple procedure which is guaranteed to 4–color any planar graph.

Task 1.11.2: Four–color the graph shown above.

Now we prove the planar graph fact used above. This is the plan: we assume that it is possible for a planar graph to have all vertices of order at least 6. Using this assumption we will end up with a nonsensical statement. This shows that the assumption was not valid.

We use Euler's formula $v - e + f = 2$. We also need the earlier observations:

Sum of the orders of all the faces = $2e$,

and

Sum of the orders of all the vertices = $2e$.

First consider the faces. Each face must have at least 3 edges. In other words, the order of each face must be ≥ 3. If we counted '3' for each face then we would get something smaller than if we counted the order of each face. In other words, $3f \leq 2e$. This can also be written $f \leq \frac{2}{3}e$. Note: a more complete version of this same argument is given in Section 1.6.

Next consider the vertices. **IF** every vertex has order at least 6, then counting '6' for each vertex gives something smaller than counting the order of each vertex. In other words, $6v \leq 2e$. This can also be written $v \leq \frac{1}{3}e$.

Here are the three relationships we have:

$$2 = v - e + f$$

$$v \leq \frac{1}{3}e$$

$$f \leq \frac{2}{3}e.$$

Putting them all together gives

$$2 \leq \frac{1}{3}e - e + \frac{2}{3}e$$

so,

$$2 \leq 0.$$

That last inequality is nonsense, so our assumption that all vertices had order at least 6 is invalid, so we conclude that at least one vertex has order 5 or less. End of Proof.

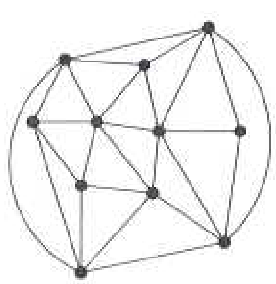
Task 1.11.3: Modify the above calculation to show that if a planar graph has all vertices of order 5 or more, then the graph must have at least 30 edges. Conclude that it must have at least 12 vertices. Find a graph with 12 vertices, each vertex having order 5.

We end this chapter by describing a procedure for 5–coloring any planar graph. This procedure is a modification of the 6–coloring method. The basis for the improvement is the following fact:

Another Planar Graph Fact. Given any 5 vertices in a planar graph, there must be two which are not directly connected to each other.

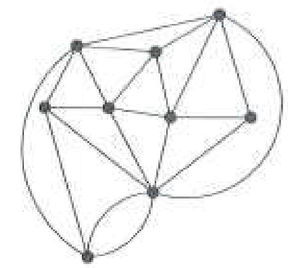
The proof is simple: If all 5 vertices were directly connected to each other then that would be K_5, but K_5 is not planar.

We use this new fact to modify Step 2 of the procedure. We delete vertices and edges as before, but then we *glue two of the vertices together*. Specifically, if there were five vertices connected to the vertex we just deleted, choose two of them which are not adjacent, and then form a new graph by gluing those two vertices together. If this gives multiple edges then delete all but one of each repeated edge. This ends the new Step 2. Here is an example using the previous graph:

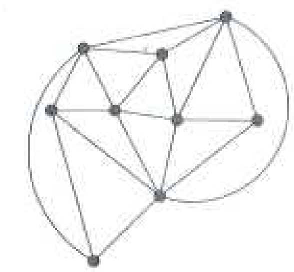


Delete a vertex of order 5 or less.

The vertices marked $*$ are not adjacent, so we can glue them together.



Combine multiple edges.



In the new procedure, repeat the new Step 2 until there are only 4 vertices left. For the new Step 3, color each of the remaining 4 vertices a different color. For the new Step 4, replace the deleted vertices in the reverse order they were removed. This is slightly more complicated than the original method because sometimes you have to rip apart two vertices which had been glued together.

To complete this Task, convince yourself that the procedure works. Note that if the graph has a vertex of order 4 or less then we can choose to delete that vertex first, so we only need to use the "gluing two vertices" step when every vertex has order 5 or more.

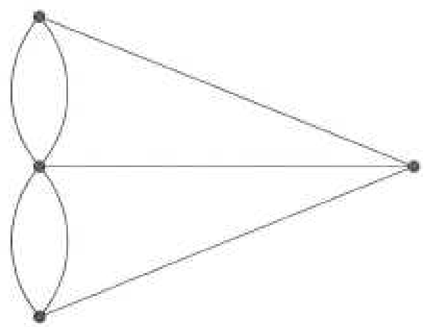
1.12 Notes

Note 1.12.a: Graphs were invented by Leonhard Euler to solve the 'Seven bridges of Königsberg' problem. The city of Königsberg, now known as Kaliningrad, Russia, had seven bridges. Here is what it looked like in Euler's time:

The citizens of the city wondered about the following question: Was it possible to take a walk which crossed each of the seven bridges exactly once?

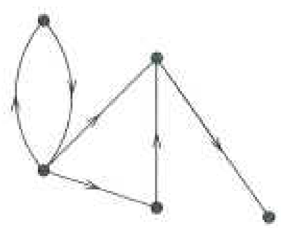
It became a common recreational activity to try and complete such a walk, but nobody succeeded in the task. Euler showed that the task was impossible. His method used the idea of an Euler path in a graph.

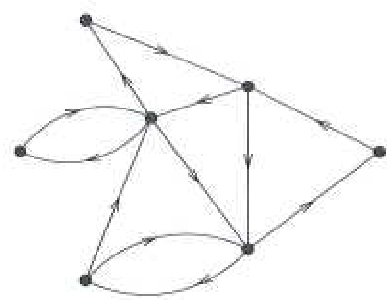
A walk crossing each bridge exactly once would correspond to an Euler path in this graph:



Each vertex corresponds to a piece of land, each edge corresponds to a bridge, and an Euler path in the graph corresponds to a walk crossing each bridge exactly once. Since the graph has four odd–order vertices, there is no Euler path, so it is impossible to walk across each of the seven bridges exactly once.

Note 1.12.b: There is a large class of graphs which we have not discussed. These are known as 'directed graphs,' commonly called **digraphs**. In a digraph the edges are like one–way streets. Here are two examples:





The main difference between a graph and a digraph is that in a digraph a path must 'follow the arrows.' For example, if you ignore the arrows then both graphs above have Euler paths. But if you must follow the arrows, then only the digraph on the right has an Euler path.

Note 1.12.c: Some people use the word **valence** where we use the word *order*.

Note 1.12.d: A graph contained within another graph is called a **subgraph**. This means that the vertices and edges of the subgraph are also vertices and edges of the larger graph. A spanning tree is a special type of subgraph.

Many properties of a graph also hold for all its subgraphs. For example, a

subgraph of a planar graph is also planar. Also, if a graph can be colored with at most N colors, then so can all its subgraphs.

Note 1.12.e: The usual way to prove Euler's formula $v - e + f = 2$ is by induction on the number of vertices and edges in the graph.

Note 1.12.f: We showed that K_5 and $K_{3,3}$ are not planar. An important result known as "Kuratowski's Theorem" says that all nonplanar graphs 'contain' either K_5 or $K_{3,3}$, or both. Exactly what it says is:

Kuratowski's theorem. Starting with any nonplanar graph, you can produce one of K_5 or $K_{3,3}$ by repeatedly performing these three moves:

Move A: Delete an edge.

Move B: Delete a vertex and all edges connected to it.

Move C: Delete a vertex of order 2, combining the two 'dangling' edges into one edge.

The idea of Kuratowski's theorem is that we can take a nonplanar graph and throw away a bunch of it until we reduce down to either K_5 or $K_{3,3}$. The proof can be found in [GT], or in any other good introductory graph theory book.

Kuratowski's theorem says that if a graph is nonplanar then it 'contains' either K_5 or $K_{3,3}$. The reverse is also true: if a graph 'contains' either K_5 or $K_{3,3}$ then it is nonplanar. Is this obvious?

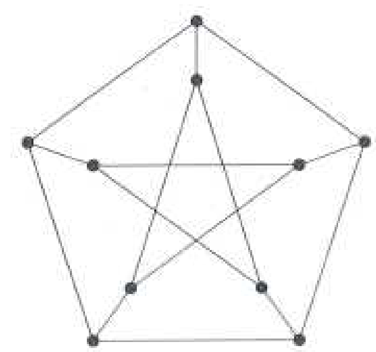
Task 1.12.1: Determine if the following statement is true:

> Suppose a graph is nonplanar, but deleting any one edge results in a graph that *is* planar. Then the original graph must be either K_5 or $K_{3,3}$.

Either explain why the statement is true, or modify the statement so that it becomes true.

Task 1.12.2: The Petersen graph is nonplanar. Draw a picture to show that the Petersen graph contains one of K_5 or $K_{3,3}$.

The Petersen graph.



Task 1.12.3: Here is an amusing way to show that the Petersen graph is nonplanar. First, show that the Petersen graph does not contain any simple closed paths with fewer than 5 edges. Second, explain why this means that any dia-

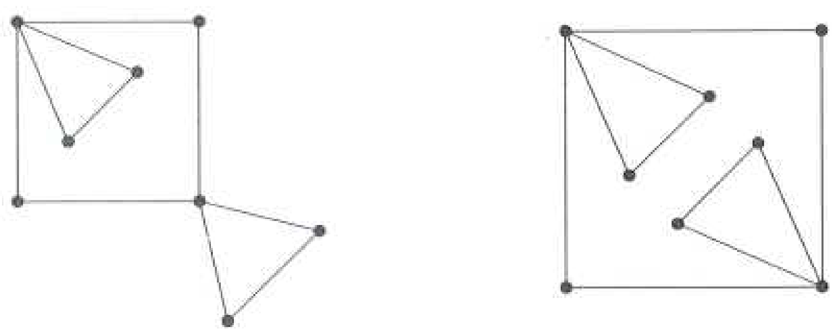
gram of the Petersen graph would have $5f \leq 2e$. Finally, use $v - e + f = 2$ and $5f \leq 2e$ to show that the Petersen graph is nonplanar.

Note 1.12.g: In this chapter we discovered that if a graph has an Euler path then it must have 0 or 2 odd order vertices. The question remains: if a graph has 0 or 2 odd order vertices, does that automatically imply that it has an Euler path? The answer is 'yes', and this can be proved by induction on the number of vertices in the graph. See [GT] or any other introductory book on graph theory.

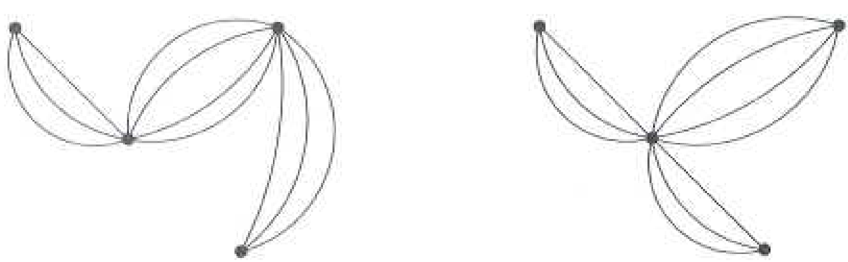
Task 1.12.4: Which graphs K_n and $K_{n,m}$ have an Euler path? Euler circuit?

Task 1.12.5: Use your answer to Task 1.12.4 to show that it is possible to place the tiles from the game of Dominoes in a circle so that the number of spots on the end of each tile matches the number of spots on the end of the adjacent tile.

Note 1.12.h: When discussing planar graphs we made the point of saying 'dual of a planar *diagram*.' This is important because we use the diagram to find the dual, and a planar graph can have several different planar diagrams. Here are two different diagrams of the same graph:



And here are the duals of those diagrams:



The duals are different: one has a vertex of order 10 and the other does not. Keep in mind that when you refer to the dual of a graph, you may need a diagram to describe which dual you mean.