sides of the same triangle, there passes a line Bp parallel to the double side bf, then this construction yields on the third side Kf of the triangle bfK, four points F, f, K, p which are in involution.

This is evident when the line BF is drawn.

When from the angle K subtended by the double side bf there passes a line KD parallel to the double side bf, this construction gives on the line FB four points F, B, D, G which are in involution, as is evident when the line KG is drawn.

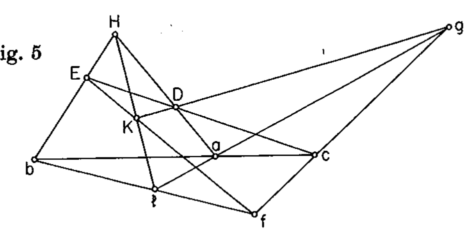
This theory swarms with similar means of deciding when four or three pairs of points are in involution on a line, but these are sufficient to open the mine of that which follows.

The last part of the *Brouillon* deals mainly with the polar properties of conics, obtained by the general intersection of a cone or cylinder (combined in the term *rouleau*, in accordance with Desargues's unifying principle, so that conics are *coupes de rouleau*).

Another outstanding contribution of Desargues's is found in an appendix of three propositions to a book published by his pupil, the master craftsman Abraham Bosse, called *Manière universelle de M. Desargues pour pratiquer la perspective* (Paris, 1648), 340–343.[4] The first contains the theorem on perspective triangles called after Desargues. The text can be found in the *Oeuvres de Desargues*, I, 430–433, or in Taton's book, pp. 206–212. The theorem is stated in the following words:

When the lines HDa, HEb, EDc, lag, flb, HKl, Dg, EKf [Fig. 5] either in different planes or in the same plane, meet in any order or direction [*biais*] whatsoever, and in similar points,[5] then the points c, f, g lie in one line cfg. For whatever form the figure takes, and in all cases, the lines being in different planes, [say] bac, lag, clb in one plane and EDc, KDg, EKf in another, then the points c, f, g are in each of the two planes, hence they are in one straight line cfg. And when the lines are in the same plane

Fig. 5



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[4] The title of this book in the *Oeuvres* of 1864 is incorrect; see Taton, p. 67.

[5] Such as Da, Eb, Kl meeting at H.

the theorem is proved by an argument involving repeated application of the theorem of Menelaus concerning a transversal in a triangle. Then the converse theorem is maintained, both for different planes and for the same plane. The argument, which is not very clear, can be studied in English translation in Smith, *Source book*, 307–311, if it is understood that Desargues's notation

$$cD - cE \begin{cases} gD - gK \\ \\ fK - fE \end{cases}$$

means $\frac{cD}{cE} = \frac{gD}{gK} \times \frac{fK}{fE}$ and expresses the application of Menelaus' theorem to transversal fcg of triangle DEK.

In a final remark Desargues observes that the figure formed by two perspective triangles in space is transformed by oblique parallel projection on the plane of one of the triangles into two perspective triangles of the plane. The two figures correspond "line to line, point to point, and reasoning to reasoning . . . and the properties of the figures can be discussed from either figure."

7 PASCAL. THEOREM ON CONICS

Blaise Pascal (1623–1662) was the son of the mathematically gifted official Etienne Pascal, and was introduced as a boy by his father to the circle of savants around Father Marin Mersenne, to which Descartes and Desargues belonged. After talking to Desargues, and probably reading his *Brouillon proiect* the 16-year-old Blaise published a "petit placard en forme d'affiche" (handbill) containing his *Essay pour les coniques* (Paris, 1640), which announces the "theorem of Pascal" concerning the so-called "hexagrammum mysticum." Like Desargues's *Brouillon*, it had only a small circulation; at present only two copies are known. Pascal uses some of the original terms of Desargues, such as *ordonnance de lignes* for *pencil of lines*.

The text is published in *Oeuvres de Pascal*, ed. L. Brunschwicg and P. Boutroux (Paris, 1908), I, 243–260, with a reproduction of the "handbill"; also in R. Taton, *L'Oeuvre mathématique de G. Desargues* (Presses Universitaires, Paris, 1951), 190–194. See also R. Taton, "L' 'Essay pour les coniques' de Pascal," *Revue d'Histoire des Sciences et de leurs Applications 8* (1955), 1–18. We base our translation on that in Smith, *Source book*, 326–330.

ESSAY ON CONICS

First Definition. When several straight lines meet at the same point, or are parallel to each other, all these lines are said to be of the same order or of the same pencil [*ordonnance*], and the totality of these lines is termed an order of lines or a *pencil* of lines.

Definition II. By the expression "conic section," we mean the circumference of the circle, the ellipse, the hyperbola, the parabola, and the rectilinear angle;

since a cone cut parallel to its base, or through its vertex, or in the three other directions which produce respectively an ellipse, a hyperbola, and a parabola, produces in the conic surface, either the circumference of a circle, or an angle, or an ellipse, a hyperbola, or a parabola.

Definition III. By the word line used alone, we mean a straight line.[1]

Lemma I. If in the plane M, S, Q [Fig. 1] two straight lines MK, MV are drawn from point M and two lines SK, SV from point S; and if K is the point

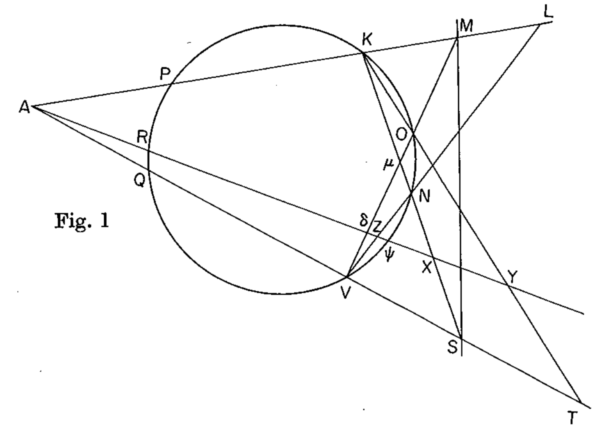


Fig. 1

of intersection of the lines MK, SK; V the point of intersection of the lines MV, SV; A the point of intersection of the lines MA, SA; and μ the point of intersection of the lines MV, SK; and if through two of the four points A, K, μ, V, which do not lie in the same line with points M, S, such as the points K, V, a circle passes cutting the lines MV, MK, SV, SK at points O, P, Q, N, then I say that the lines MS, NO, PQ are of the same order.[2]

Lemma II. If through the same line several planes are passed, and are cut by another plane, all lines of intersection of these planes are of the same order as the line through which these planes pass.

On the basis of these two lemmas and several easy deductions from them, we can demonstrate that if the same things are granted as for the first lemma, that is, through points K, V, any conic section whatever passes cutting the lines

[1] "Par le mot de droite mis seul, nous entendons ligne droite." The first definition establishes the projective equivalence of intersection and parallel lines, and is thus based on Desargues's concept of points at infinity. Similarly, the second definition establishes that any plane intersection of an (oblique) cone is a conic section. Pascal, again following Desargues, breaks with the Apollonian tradition of the "triangle par l'axe," in which the only sections of an (oblique) cone considered are in planes perpendicular to the triangle formed by a plane through the "axis."

[2] Remembering that "are of the same order" means "pass through the same finite or infinite point," we recognize "Pascal's theorem" for a circle. Notice that for Pascal his theorem is only a lemma, which indeed it was; he used it to find many other properties. We can only conjecture how Pascal proved his lemma, but it is likely that, like Desargues, for his theorem, he used the theorem of Menelaus.

MK, MV, SK, SV in points P, O, N, Q, then the lines MS, NO, PQ will be of the same order. This constitutes a third lemma.[3]

By means of these three lemmas and certain deductions therefrom, we propose to give a complete text on the elements of conics,[4] that is to say, all the properties of diameters and other straight lines, of tangents, etc., the construction of the cone from substantially these data, the description of conic sections by points, etc.

Having done this, we shall state the properties which follow, doing this in a more general manner than usual. Take, for example, the following: If in the plane MSQ, in the conic PKV, there are drawn the lines AK, AV, cutting the conic in points P, K, Q, V, and if from two of these four points that do not lie in the same line with point A—say the points K, V—and through two points N, O, taken on the conic, there are produced four lines KN, KO, VN, VO, cutting the lines AK, AV, at points L, M, T, S, then I maintain that the proportion composed of the ratios of the line PM to the line MA, and of the line AS to the line SQ, is the same as the proportion composed of the ratio of the line PL to the line LA, and of the line AT to the line TQ.[5]

We shall also demonstrate [Fig. 2] that if there are three lines DE, DG, DH that are cut by the lines AP, AR at points F, G, H, C, γ, B and if the point E be

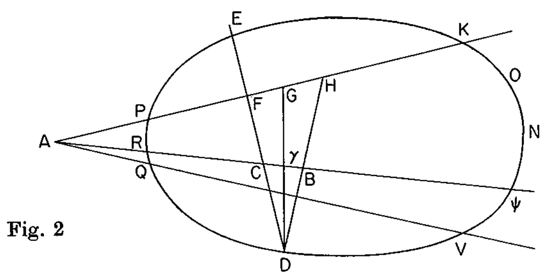


Fig. 2

fixed in the line DC, [a] the proportion composed of the ratios of the rectangle $EF.FG$ to the rectangle $EC.C\gamma$, and of the line $A\gamma$ to the line AG, is the same as the ratio of the rectangle $EF.FH$ to the rectangle $EC.CB$, and of the line AB to the line AH.[6] [b] The same is also equal to the ratio of the rectangle $FE.FD$ to the rectangle $CE.CD$. Consequently, if a conic section passes through

[3] This is the extension of Pascal's theorem to any conic, including the degenerate case (hence "Pascal's theorem" in modern axiomatics, the special case already known to Pappus).

[4] We know that Pascal (*Oeuvres*, II, 220) worked on this treatise, and Leibniz in a letter of 1676 reported on his study of the manuscript. After that time it disappeared.

[5] This theorem is equivalent to the statement that the cross ratios $(ALMP)$ and $(ASTQ)$ are equal.

[6] [a] $\dfrac{EF \times FG}{EC \times C\gamma} \times \dfrac{A\gamma}{AG} = \dfrac{EF \times FH}{EC \times CB} \times \dfrac{AB}{AH}.$

This is Pappus' theorem; in modern terms: the four rays of the pencil $D(A, C, \gamma, B)$ cut out from two intersecting lines point ranges with equal cross ratios (the term "cross ratio" is due to W. K. Clifford, 1869).

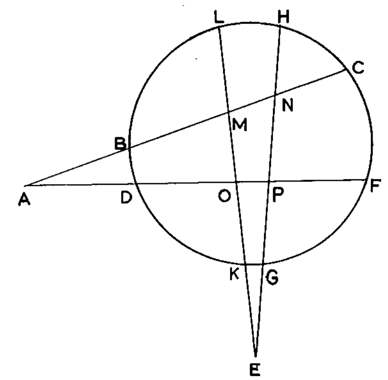
[b] This ratio $\dfrac{EF \times FH}{EC \times CB} \times \dfrac{AB}{AH}$ is also equal to $\dfrac{FE \times FD}{CE \times CD}$.

(footnote continued)

the points E, D, cutting the lines AH, AB in points P, K, R, ψ, [c] the proportion composed of the ratio of the rectangle of these lines EF, FG, to the rectangle of the lines EC, $C\gamma$, and of the line γA to the line AG, will be the same as the ratio of the rectangle of the lines FK, FP to the rectangle of the lines CR, $C\psi$, and of the rectangle of the lines AR, $A\psi$ to the rectangle of the lines AK, AP.

We can also show that if four lines AC, AF, EH, EL [Fig. 3] intersect in points N, P, M, O, and if a conic section cuts these lines in points C, B, F, D,

Fig. 3



H, G, L, K, the proportion consisting of the ratios of rectangle $MC.MB$ to rectangle $PF.PD$, and of rectangle $AD.AF$ to rectangle $AB.AC$, is the same as the proportion composed of the ratios of rectangle $ML.MK$ to the rectangle $PH.PG$, and of rectangle $EH.EG$ to rectangle $EK.EL$.[7]

We can also demonstrate a property stated below, due to M. Desargues of Lyons, one of the great minds of this time and one of the best versed in mathematics, particularly in conics, whose writings on this subject, although few in number, give abundant proof of his knowledge to those who seek for information. I should like to say that I owe the little that I have found on this subject to his writings, and that I have tried to imitate his method, as far as possible, in which he has treated the subject without making use of the triangle through the axis.[8]

Giving a general treatment of conic sections, the following is the remarkable property under discussion: If in the plane MSQ [Fig. 1] there is a conic section

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This can be reduced to Menelaus' theorem for the triangle:

$$FH \times AB \times CE = AH \times CB \times DF,$$

also used by Desargues.

[c] $$\frac{EF \times FG}{EC \times C\gamma} \times \frac{\gamma A}{AG} = \frac{FK \times FP}{Cr \times C\psi} = \frac{Ar \times A\psi}{AK \times AP}.$$

The last identity expresses a relation between the segments that the conic intercepts on the sides of the triangle AFC.

7 $$\frac{MC \times MB}{PF \times PD} \times \frac{AD \times AF}{AB \times AC} = \frac{ML \times MK}{PH \times PG} \times \frac{EH \times EG}{EK \times EL}.$$

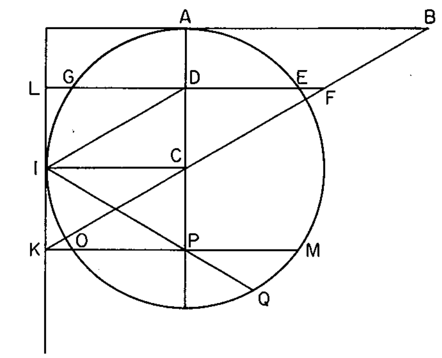
This formula establishes the relation between the segments a conic section cuts out on the sides of quadrilateral $MNOP$.

8 See note 1.

PQN, on which are taken four points K, N, O, V from which are drawn the lines KN, KO, VN, VO, in such a way that through the same four points only two lines may pass, and if another line cuts the conic at points R, ψ, and the lines KN, KO, VN, VO in points X, Y, Z, δ, then as the rectangle $ZR.Z\psi$ is to the rectangle $\gamma R.\gamma\psi$, so the rectangle $\delta R.\delta\psi$ is to the rectangle $XR.X\psi$.[9]

We can also prove that if in the plane of the hyperbola, the ellipse, or the circle AGE [Fig. 4] of which the center is C the line AB is drawn touching the

Fig. 4



section at A, and if having drawn the diameter we take line AB such that its square shall be equal to one-fourth of the rectangle of the figure and if CB is drawn, then for any line such as DE, parallel to line AB and cutting the section in E and the lines AC, CB in points D, F, the property holds that if the section AGE is an ellipse or a circle, the sum of the squares of the lines DE, DF will be equal to the square of the line AB;[10] and in the hyperbola, the difference between the same squares of the lines DE, DF will be equal to the square of the line AB.

We can also deduce from this several problems; for example:

From a given point to draw a tangent to a given conic section;

To find two diameters that meet at a given angle;

To find two diameters cutting at a given angle and having a given ratio.

There are many other problems and theorems, and many deductions which can be made from what has been stated above, but the distrust which I have owing to my little experience and capacity, does not allow me to go further into the subject until it has passed the examination of able men who may be willing to take this trouble. After that if someone thinks the subject worth continuing, we shall endeavor to extend it as far as God gives us the strength.

At Paris, M.DC.XL.

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9 This is Desargues's theorem on the involution cut out by a pencil of conics on a transversal.

10 $AB^2 = DE^2 + DF^2$, equal to the square of the minor axis, determines the ellipse. This equality is equivalent to the equation of the ellipse in its modern form, $x^2/a^2 + y^2/b^2 = 1$, and is related to propositions of Apollonius and Desargues. In the case of a circle, "the fourth of the rectangle of the figure" is the square of the radius.