In the choice of the selections, especially those of Euler, we had to be fairly arbitrary. We decided to concentrate on power residues. The reader who wants to know more about eighteenth-century number theory should consult M. Cantor, *Vorlesungen über Geschichte der Mathematik*, vol. III (Teubner, Leipzig, 1898) and especially, in vol. IV (1908), the article by F. Cajori. Further details can be found in the three volumes of L. E. Dickson, *History of the theory of numbers* (Carnegie Institution, Washington, D.C., 1919–1923; 2d ed., 1934).

Among the problems solved in the *Liber abaci* one has acquired considerable fame. It may well have been invented by Leonardo, and with it we open our collection of texts.

1 LEONARDO OF PISA. THE RABBIT PROBLEM

In Leonardo of Pisa, also called Fibonacci, we meet the first outstanding mathematician of the Latin Middle Ages. He was a merchant of Pisa who traveled widely in the world of Islam, and took the opportunity of studying Arabic mathematical writings. His work is in the spirit of the Arabic mathematics of his day, but also reveals his own position as an independent thinker. Leonardo's *Liber abaci* (1202, revised 1228) circulated widely in manuscript, but was published only in 1857: *Scritti di Leonardo Pisano* (pubbl. da B. Boncompagni, Tipografia delle scienze matematiche e fisiche, Rome; 2 vols., 459 pp.). It is a voluminous compendium on arithmetic and its mercantile practice (even finger counting), the theory of linear, quadratic, and simultaneous sets of equations, square and cube roots. One of the principal features of the book is that, from the first page on, Leonardo introduces and uses the decimal position system. The first chapter opens with the sentence: "These are the nine figures of the Indians

9 8 7 6 5 4 3 2 1

With these nine figures, and with this sign 0 which in Arabic is called zephirum,[1] any number can be written, as will below be demonstrated."

We confine ourselves here to presenting, in translation from the Latin, two interesting sections from the *Liber abaci* that may very well be original contributions. Since the first introduces *paria coniculorum*, we know it as the rabbit problem. It stands by itself (vol. I, 283–284), sandwiched in between other problems; the one before it deals with the so-called perfect numbers 6, 28, 496, . . . , and the one after with the solution of a system of four linear equations with four unknowns.

How many pairs of rabbits can be bred from one pair in one year?

A man has one pair of rabbits at a certain place entirely surrounded by a wall. We wish to know how many pairs can be bred from it in one year, if the nature of these rabbits is such that they breed every month one other pair and begin to

[1] *Zephirum*, *zephyr*, from Arabic *as-sifr*, literal translation of Sanskrit *sunya* = empty, ... in its turn led to French *chiffre*, German *Ziffer*, and English *zero* and *cipher*. The mean-

breed in the second month after their birth. Let the first pair breed a pair in the first month, then duplicate it and there will be 2 pairs in a month. From these pairs one, namely the first, breeds a pair in the second month, and thus there are 3 pairs in the second month. From these in one month two will become pregnant, so that in the third month 2 pairs of rabbits will be born. Thus there are 5 pairs in this month. From these in the same month 3 will be pregnant, so that in the fourth month there will be 8 pairs. From these pairs 5 will breed 5 other pairs, which added to the 8 pairs gives 13 pairs in the fifth month, from which 5 pairs (which were bred in that same month) will not conceive in that month, but the other 8 will be pregnant. Thus there will be 21 pairs in the sixth month. When we add to these the 13 pairs that are bred in the 7th month, then there will be in that month 34 pairs . . . [and so on, 55, 89, 144, 233, 377, . . .]. Finally there will be 377. And this number of pairs has been born from the first-mentioned pair at the given place in one year. You can see in the margin how we have done this, namely by combining the first number with the second, hence 1 and 2, and the second with the third, and the third with the fourth . . . At last we combine the 10th with the 11th, hence 144 with 233, and we have the sum of the above-mentioned rabbits, namely 377, and in this way you can do it for the case of infinite numbers of months.[2]

	pairs
	1
first	2
second	3
third	5
fourth	8
fifth	13
sixth	21
seventh	34
eighth	55
ninth	89
tenth	144
eleventh	233
twelfth	377

Here is a section (I, 24) in which Leonardo introduces a kind of continued fraction, which he writes $\frac{e\,c\,a}{f\,d\,b}$, or in our notation:

$$\frac{e\,c\,a}{f\,d\,b} = \cfrac{a + \cfrac{c + \frac{e}{f}}{d}}{b} = \frac{adf + cf + e}{bdf} = \frac{a}{b} + \frac{c}{d}\cdot\frac{1}{b} + \frac{e}{f}\cdot\frac{1}{b}\cdot\frac{1}{d}.$$

Below some line [branchlet, *virgula*] let there be 2, 6, 10, and above the 2 be 1, above 6 be 5, and above 10 be 7, which appears as $\frac{1\ 5\ 7}{2\ 6\ 10}$. The 7 above 10 at the head of the line represents seven-tenths and the 5 above 6 denotes five-sixths

[2] This sequence of numbers, 1, 2, 3, 5, 8, . . . , u_n, . . . , with the property that $u_n = u_{n-1} + u_{n-2}$, $u_0 = 1$, $u_1 = 1$, is called a *Fibonacci series*. It has been the subject of many investigations, and is closely connected with the golden section, that is, the division of a line segment AB by a point P such that $AP : AB = PB : AP$. See, for example, R. C. Archibald, "The golden section," *American Mathematical Monthly 25* (1918), 232–238; D'Arcy W. Thompson, *On growth and form* (Cambridge University Press, New York, 1942), 912–933; H. S. M. Coxeter, "The golden section, phyllotaxis, and Wythoff's game," *Scripta Mathe-*