Last conclusion: A difformly difform quality is represented by a ... which the line representing the subject is the base, while the summit ... nonstraight line, not parallel to the base. Such a difformity may be ... an infinite number of different ways, for the summit line may vary ... many ways.

However, someone might say: It is not necessary to represent a quality ... way. I say that the representation is good, as also appears in Aristotle[4] ... represents time by a line. In the same way in *Perspectiva* the *virtus* ... represented by a triangle.[5] Moreover according to this representation ... understand more easily what is said about uniformly difform qualities ... consequently the representation is good.

This means that, since qualities are represented by surfaces, the equality of two ... may also be transferred to the qualities which they represent. In this case, there ... has to prove that surface $OCBA$ = surface $OCED$, and from this equality it then ... that the uniformly difform quality that is represented by $OCBA$ is equal to the ... quality that is represented by $OCED$.

2 REGIOMONTANUS. TRIGONOMETRY

Trigonometry was developed into a independent branch of mathematics by Islamic ... notably by Nasir ed-dīn at-Tūsī (or Nasir Eddin, 1201–1274). The first publication in ... Europe to achieve the same goal was Regiomontanus' *De triangulis omnimodis* (On ... of all kinds; Nuremberg, 1533).

[4] Aristotle, *Physica*, IV, 11; 220a4–20. In lines 219b1–2 Aristotle defines time as "... motus secundum prius et posterius." Here he tries to explain that the "now-moment ... the one hand, divides time into two parts (past–future), but, on the other hand, ... continuous. He compares time to a line on which a point makes a division but ... stitutes continuity on the line.

[5] The *virtus activa* is the light diffused from the source of light (*lumen*). Later, in ... 17, Oresme concludes: "Such a force or such a light extends uniformly difformly; ... words: it is a uniformly difform quality. This appears plausible because—since the ... does not extend uniformly—it seems to diminish as the distance increases; this ... has to take place proportionally, i.e., uniformly difformly" [Fig. 3]. The *Perspectiva* ... tioned is the one written by Witelo (Vitellio), a Polish mathematician of the ...

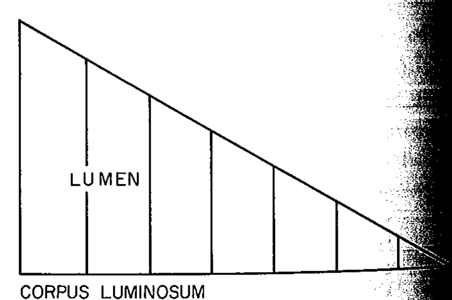
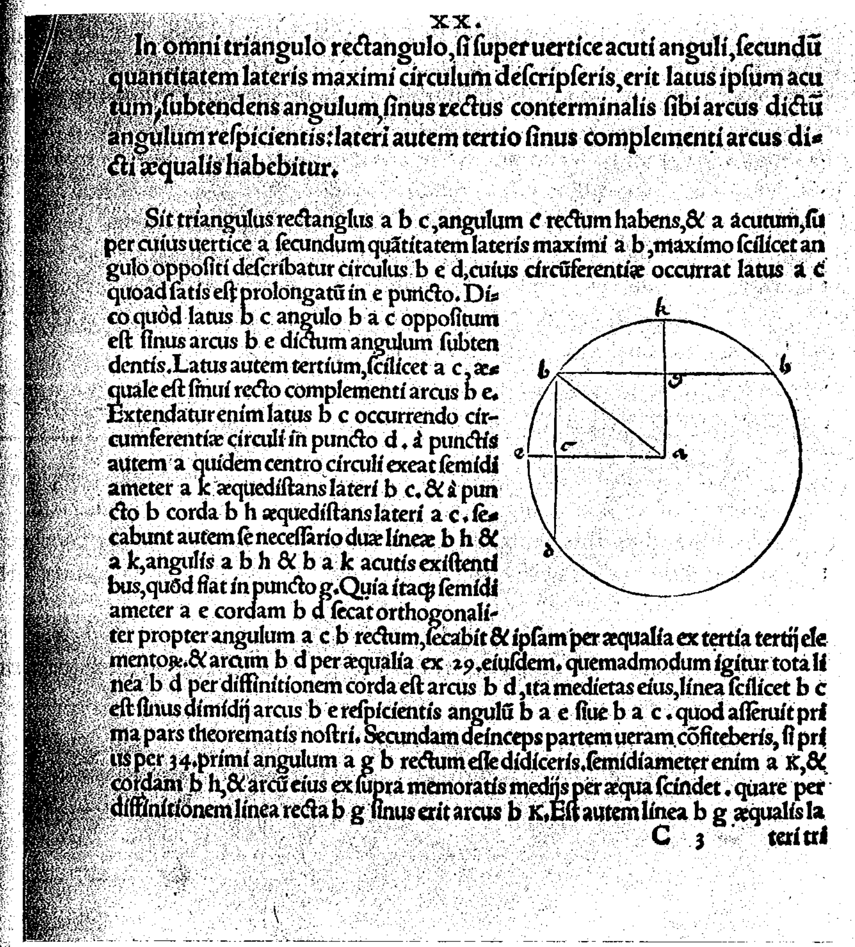


Fig. 3

century, first printed in Nuremberg (1535), a book that was widely read, and ... Kepler wrote a book, *Ad Vitellionem paralipomena* (Frankfurt a. M., 1604).



XX.

In omni triangulo rectangulo, si super uertice acuti anguli, secundũ quantitatem lateris maximi circulum descripseris, erit latus ipsum acutum, subtendens angulum, sinus rectus conterminalis sibi arcus dictũ angulum respicientis: lateri autem tertio sinus complementi arcus dicti æqualis habebitur.

Sit triangulus rectanglus a b c, angulum c rectum habens, & a acutum, super cuius uertice a secundum quãtitatem lateris maximi a b, maximo scilicet angulo oppositi describatur circulus b e d, cuius circũferentiæ occurrat latus a c quoad satis est prolongatũ in e puncto. Dico quòd latus b c angulo b a c oppositum est sinus arcus b e dictum angulum subtendentis. Latus autem tertium, scilicet a c, æquale est sinui recto complementi arcus b e. Extendatur enim latus b c occurrendo circumferentiæ circuli in puncto d. à punctis autem a quidem centro circuli exeat semidiameter a k æquedistans lateri b c. & à puncto b corda b h æquedistans lateri a c. secabunt autem se necessario duæ lineæ b h & a k, angulis a b h & b a k acutis existentibus, quòd fiat in puncto g. Quia itaq; semidiameter a e cordam b d secat orthogonaliter propter angulum a c b rectum, secabit & ipsam per æqualia ex tertia tertij elementorum. & arcum b d per æqualia ex 29. eiusdem. quemadmodum igitur tota linea b d per diffinitionem corda est arcus b d, ita medietas eius, linea scilicet b c est sinus dimidij arcus b e respicientis angulũ b a e siue b a c. quod asseruit prima pars theorematis nostri. Secundam deinceps partem ueram cõfiteberis, si prius per 34. primi angulum a g b rectum esse didiceris. semidiameter enim a K, & cordam b h, & arcũ eius ex supra memoratis medijs per æqua scindet. quare per diffinitionem linea recta b g sinus erit arcus b K. Est autem linea b g æqualis lateri tri

C 3

Fig. 1

Johannes Müller (1436–1476), called Regiomontanus from his birthplace, Königsberg in Franconia, was an instrument maker, mathematician, astronomer, and humanist who settled at Nuremberg and died in Rome as adviser to the pope on calendar reform. His trigonometry, finished in 1464, remained in manuscript until 1533. The book, reprinted at Basel in 1561, was widely studied during the sixteenth century. It deals with both plane and spherical trigonometry without using formulas: all theorems and demonstrations are verbal, with frequent references to Euclid's *Elements*. The trigonometric concepts used are the sine (*sinus* or *sinus rectus*) and versed sine (*sinus versus*), conceived as line segments and expressed as parts of a given radius (*sinus totus*) $R = 60.000$.[1]

The English translation we give of parts of the book is based on that by B. Hughes in *Regiomontanus on triangles* (University of Wisconsin Press, Madison, 1967), 59. First our text leads up to the sine law of plane triangles; then we present Regiomontanus' way of stating the cosine law for spherical triangles. In the figures capital letters $A, B, \ldots$ are used instead of Regiomontanus' $a, b, \ldots$; see Fig. 1.

Book I. *Theorem* 20. In every right triangle, if we describe a circle with center a vertex of an acute angle and radius the length of the longest side,[2] then the side subtending this acute angle is the right sine [*sinus rectus*] of the arc adjacent to that side and opposite the given angle; the third side is equal to the sine of the complement of the arc.

If a right triangle ABC [Fig. 2] is given with C the right angle and A an acute angle, around the vertex of which a circle BED is described with the longest

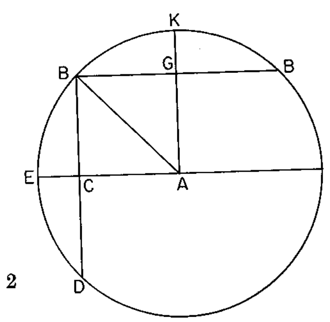


Fig. 2

side—that is, the side opposite the largest angle—as radius, and if side AC is extended sufficiently to meet the circumference of the circle at point E, then side BC opposite angle BAC is the sine of arc BE subtending the given angle, and furthermore the third side AC is equal to the right sine of the complement of arc BE.

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[1] Hence Regiomontanus' *sinus* or *sinus rectus* of an angle α is our $R \sin \alpha$, and his *sinus versus* of an angle α is our R versin $\alpha = R(1 - \cos \alpha)$. See Selection I.4, note 2.

[2] Regiomontanus does not use the term "hypotenuse," herein following Euclid. Neither does he use the term "trigonometry," which appears first in the title of the book of Bartolomeus Pitiscus, *Trigonometria* (Heidelberg, 1595).

Then, extending BC to CD, just as by definition the entire line BD is the chord of arc BD, so also its half, namely line BC, is the sine of the half-arc BE opposite angle BAE or BAC.

Book I. *Theorem* 28. When the ratio of two sides of a right triangle is given, its angles can be ascertained.

One of the two sides is opposite the right angle or else none is. First, if side AB, whose ratio to side AC is known, is opposite right angle ACB, then the angles of this triangle become known.

For instance, if in triangle ABC (Fig. 1), $AB:BC = 9:7$, then multiply 7 into $R = 60.000$ (the whole right sine, *sinus rectus totus*) and divide by 9. The quotient, 46667, corresponds to about 51°3′, the value of angle ABC.[3]

Book II opens with the sine law for plane triangles.

Book II. *Theorem* 1. In every rectilinear triangle the ratio of one side to another side is as that of the right sine of the angle opposite one of the sides to the right sine of the angle opposite the other side.

As we said elsewhere, the sine of an angle is the sine of the arc subtending that angle. Moreoever, these sines must be related through one and the same radius of the circle or through several equal radii. Thus, if triangle ABG [Fig. 3] is a

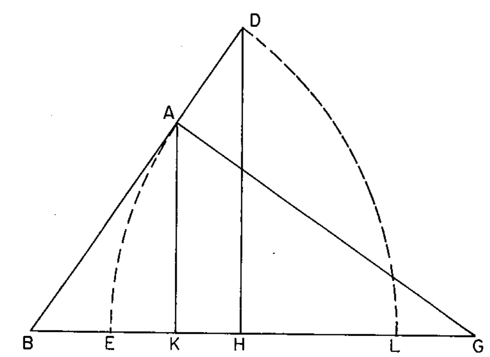

Fig. 3

rectilinear triangle, then the ratio of side AB to side AG is as that of the sine of angle AGB to the sine of angle ABG; similarly, that of side AB to BG is as that of the sine of angle AGB to the sine of angle BAG.

[3] Regiomontanus had sine tables for several values of R, which he may have computed himself, or taken from other, perhaps Arabic, sources. The table for $R = 60.000$ was first published in the *Tabulae directionum projectionumque* (Augsburg, 1490). The Basel edition of *De triangulis* also has a table.

If triangle ABG is a right triangle, we will provide the proof directly from Theorem I.28 above. However, if it is not a right triangle yet the two sides AB and AG are equal, the two angles opposite the sides will also be equal and hence their sines will be equal. Thus from the two sides themselves it is established that our proposition is verified. But if one of the two sides is longer than the other—for example, if AG is longer—then BA is drawn all the way to D, until the whole line BD is equal to side AG. Then around the two points B and G as centers, two equal circles are understood to be drawn with the lengths of lines BD and GA as radii respectively. The circumferences of these circles intersect the base of the triangle at points L and E, so that arc DL subtends angle DBL, or ABG, and arc AE subtends angle AGE, or AGB. Finally two perpendiculars AK and DH, from the two points A and D, fall upon the base. Now it is evident that DH is the right sine of angle ABG and AK is the right sine of angle AGB. Moreover, by VI.4 of Euclid,[4] the ratio of AB to BD, and therefore to AG, is as that of AK to DH. Hence what the proposition asserts is certain.

Then follow many applications; for instance, Theorem 2 shows how to find the sides of a triangle if their sum is known together with the angles opposite them.

Book V. *Theorem* 2. In every spherical triangle that is constructed from the arcs of great circles, the ratio of the versed sine of any angle to the difference of two versed sines, of which one is the versed sine of the side subtending this angle while the other is the versed sine of the difference of the two arcs including this angle, is as the ratio of the square of the whole right sine to the rectangular product of the sines of the arcs placed around the mentioned angle.

In this theorem we recognize, in geometric and hence homogeneous form, the cosine law for a spherical triangle. We omit the proof, which is quite complicated. In our notation:

$$\frac{R \text{ versin } \alpha}{R \text{ versin } a - R \text{ versin } (b-c)} = \frac{R^2}{R \sin b \cdot R \sin c},$$

where a, b, c are the sides and α is the angle opposite a in the spherical triangle on a sphere of radius R. The expression can be reduced to

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha.$$

[4] Theorem 4 of Book VI of Euclid's *Elements* states that in similar triangles corresponding sides are proportional.

3 FERMAT. COORDINATE GEOMETRY

Analytic geometry (the term itself, in its present meaning, appears first in the beginning of the nineteenth century) can be dated back to the works on coordinate geometry by Descartes (1637; Selections II.7, 8) and Fermat. Fermat's papers, probably written about the same time as Descartes's work, were posthumously published by his son in *Varia opera mathematica* (Toulouse, 1679), and thus had less influence than the work of his rival. Both authors were moved by the same spirit: they wanted to show how the Renaissance algebra of Cardan and his successors could be applied to the geometry of the Greeks, notably to Apollonius' theory of loci as preserved by Pappus. In carrying out their program they differed in their methods. Fermat used the sixteenth-century notation of Viète, in which, as we have seen, our $Dx = By$ is written "D in A aequatur B in E," and in which the homogeneity of the formulas is preserved: when D and A represent line segments, then "A in D aeq. Z pl" stands for "A times D is equal to the area Z (Z plane)." Descartes introduced the notation still in use in which known constant quantities are indicated by the letters $a, b, \ldots$, unknown or variable quantities by $x, y, \ldots$, their squares, cubes, and so on by $aa = a^2, a^3, \ldots, xx = x^2, x^3$, and so on. Descartes also rejected the homogeneity of the formulas (see Selections III.4, 5).

Descartes's discussion consists in giving examples of his method. Fermat, starting with loci expressed by straight lines and following these with loci expressed by conic sections, has a method that shows some similarity with our way of introducing analytic geometry.

Both Descartes and Fermat used as an important test case for their methods the so-called problem of Pappus, found in Book VII of Pappus' *Collection* (*Synagōgē*), written at about the end of the third century A.D. On this problem see M. R. Cohen and I. E. Drabkin, *A source book in Greek science* (Harvard University Press, Cambridge, 1948), 79–80, and T. L. Heath, *History of Greek mathematics* (Clarendon Press, Oxford, 1921), II, 400–401.[1] Here follows Pappus' text, which is preceded by a remark that Apollonius, in the third book of his *Conics* (c. 220 B.C.), mentions "the locus for three and four lines." Then Pappus continues:

"But this locus of three and four lines, of which Apollonius says, in his third book, that Euclid has not treated it completely, he himself has also not been able to achieve, and he has not even been able to add anything to what Euclid has written about it . . .

"Here we shall show what is that locus of three and four lines . . . Let three straight lines be given in position. Let there pass through the same point to these three straight lines three others at given angles, and let the ratio of the rectangle taken on two of these lines to the square of the third be given. Then the point will be on a solid line given in position, that is, on one of the three conics.[2] And if one passes straight lines at given angles to four straight lines given in position, and if the ratio of the rectangle taken on two of them to that taken on the other two is given, then the point will also be on a conic section given in position. On the other hand, if there are only two straight lines, then it is known that the locus is plane, but if there are more than four lines, then the locus of the point is no longer

[1] Pappus' *Collection* can be consulted in the French translation by P. Ver Eecke: *Pappus d'Alexandrie. La collection mathématique* (2 vols.; Declès de Brouwer, Paris, Bruges, 1933; Paris, 1959). The quoted text is on pp. 507–510.

[2] Pappus distinguishes between plane, solid, and linear problems. The plane problems require only circles and straight lines for their construction, the solid ones require general conic sections, and the linear ones require more complex curves. This distinction is taken over by Fermat as well as by Descartes (see Selection III.4).