Twin Paradox from Every Angle

The Scenario: Goslo (unprimed) on Earth, Speedo (primed) travels at 0.5c to planet X, which is 10cy away, and then returns (double primed). Goslo sends a light message every year. Subscripts: 1=trip out, 2=return trip, 0=proper quantity, s is about sending messages, r is about receiving them.

Goslo's perspective, traveling out

a) The outbound trip takes $t_1 = \frac{L_0}{v} = \frac{10c\text{y}}{0.5c} = 20\text{y}$.

b) Goslo sends pulses every $T_{s0} = 1\text{y}$. He will be sending out 40 pulses total.

c) When multiple messages are "in flight," they are spaced apart by a distance $d_1 = cT_{s0} = 1c\text{y}$. With no object to measure, this is not a proper length.

d) Since the light has to catch up to Speedo, messages are received less frequently than sent. Call the receiving interval T_{r1}.

1. Starting at receipt of one message, the next message has to travel d_1 ***plus*** the extra distance that Speedo covers. Message travels $x_1 = d_1 + vT_{r1}$.
2. The time required for the light message to do that is

$$T_{r1} = \frac{x_1}{c} = \frac{d_1 + vT_{r1}}{c} = \frac{cT_{s0} + vT_{r1}}{c} = T_{s0} + \beta T_{r1}, \text{ so } T_{r1} = \frac{T_{s0}}{1-\beta} = 2T_{s0} = 2\text{y}.$$

e) During outbound flight, Speedo receives $N_1 = \frac{t_1}{T_{r1}} = \frac{20\text{y}}{2\text{y}} = 10$ pulses.

When Speedo is at planet X, 10 more messages will be in flight.

Note T_{s0} is a proper time, T_{r1} is not. Speedo measures proper time between receipts.

Goslo's perspective, returning

f) Return trip takes $t_2 = 20\text{y}$. Messages are still sent out every $T_{s0} = 1\text{y}$, and in-flight messages are still spaced apart by $d_2 = 1c\text{y}$.

g) Since Speedo is approaching, messages are received more frequently.

1. Starting at receipt of one message, the next message has to travel d_2 ***less*** the extra distance that Speedo covers. Message travels $x_2 = d_2 - vT_{r2}$.
2. So $T_{r2} = \frac{x_2}{c} = \frac{d_2 - vT_{r2}}{c} = \frac{cT_{s0} - vT_{r2}}{c} = T_{s0} - \beta T_{r2}$, and $T_{r2} = \frac{T_{s0}}{1+\beta} = \frac{2}{3}T_{s0} = \tfrac{2}{3}\text{y}$
.

h) On return flight, Speedo receives $N_2 = \frac{t_2}{T_{r2}} = \frac{20\text{y}}{\tfrac{2}{3}\text{y}} = 30$ pulses.

Great, so all pulses are received.

Speedo's perspective, traveling out

a) The travel distance is only $L' = \frac{1}{\gamma}L_0 = \frac{10c\text{y}}{1.155} = 8.66c\text{y}$. So the outbound trip takes $t'_{10} = \frac{L'}{v} = \frac{8.66c\text{y}}{0.5c} = 17.32\text{y}$. (That's a proper time.)

b) Goslo's clock is running slow, so he emits a pulse every $T'_s = \gamma T_{s0} = 1.155\text{y}$

c) Spacing of in-flight messages is big, since Earth recedes between sendings.

1. In the time between sending two messages, the first message travels cT'_s while the Earth travels vT'_s in the opposite direction.
2. So they are spaced apart a distance of $d'_1 = cT'_s + vT'_s = (1+\beta)cT'_s = 1.732c\text{y}$

d) Once in flight, messages simply approach Speedo at c, so the time between receipts is $T'_{r10} = \frac{d'_1}{c} = (1+\beta)T'_s = 1.732\text{y}$

e) During outbound flight, Speedo receives this many pulses:

$$N_1 = \frac{t'_1}{T'_{r10}} = \frac{(L_0/\gamma v)}{(1+\beta)\gamma T_{s0}} = \frac{L_0}{(1+\beta)\gamma^2 vT_{s0}} = \frac{1-\beta^2}{1+\beta}\frac{L_0}{vT_{s0}} = (1-\beta)\frac{L_0}{vT_{s0}} = 0.5\frac{10c\text{y}}{0.5c(1\text{y})} = 10$$

So everyone agrees! Note that d'_1 and d_1 are both "the distance between in-flight messages," but **neither** is a proper length, because there is no object to measure.

Speedo's perspective, returning

This is a new, third frame, so I'll use double-primes. However, its speed is the same as the speed of the primed frame.

f) Return trip takes $t''_2 = 17.32\text{y} = t'_1$, messages are sent every $T''_s = 1.155\text{y} = T'_s$

g) In-flight message spacing is extra small, as Earth approaches between sendings.

1. In the time between sending two messages, the first message and Earth travel the same distances as before (cT''_s and vT''_s), but now the same direction.
2. So the messages are spaced apart a distance of $d''_2 = cT''_s - vT''_s = (1-\beta)cT''_s = 0.577c\text{y}$.

h) The time between receipts is $T''_{r20} = \frac{d''_2}{c} = (1-\beta)T''_s = 0.577\text{y}$.

i) During return flight, Speedo receives this many pulses:

$$N_2 = \frac{t''_{20}}{T''_{r20}} = \frac{L_0/\gamma v}{(1-\beta)\gamma T_{s0}} = \frac{1-\beta^2}{1-\beta}\frac{L_0}{vT_{s0}} = (1+\beta)\frac{L_0}{vT_{s0}} = 1.5\frac{10c\text{y}}{0.5c(1\text{y})} = 30$$

So again, everyone agrees. Note: when the message spacing changes (during Speedo's turn around) from d'_1 to d''_2, some big changes have to happen.