Oscillations

Simple Harmonic Oscillator (SHO)

Newton's Second Law: $-kx = m\ddot{x}$

Equations of Motion: $\ddot{x} + {\omega_0}^2 x = 0$, where $\omega_0 = \sqrt{\frac{k}{m}}$

Solution: $x(t) = A\cos(\omega_0 t + \phi)$

Useful Equation: $\omega = 2\pi f = \frac{2\pi}{T}$

Damped Oscillations

Newton's Second Law:

$$-kx - bv = m\ddot{x}$$

Equations of Motion:

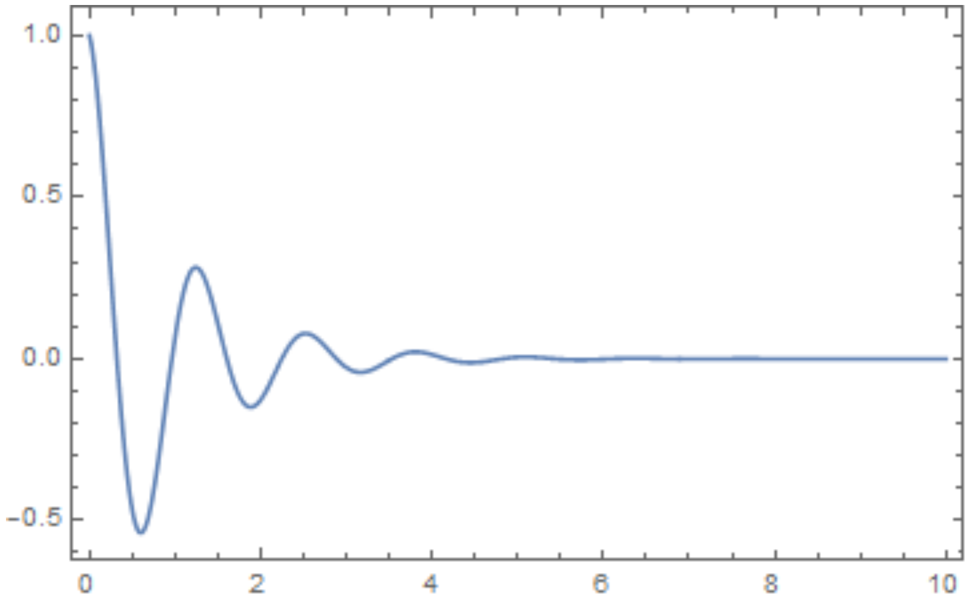
$$\ddot{x} + 2\beta\dot{x} + {\omega_0}^2 x = 0\text{, where } \omega_0 = \sqrt{\frac{k}{m}} \text{ and } \beta = \frac{b}{2m}$$

Underdamped: $\beta < \omega_0$

Solution:

$$x(t) = Ae^{-\beta t}\cos(\omega_1 t + \phi)$$

Where $\omega_1 = \sqrt{{\omega_0}^2 - \beta^2}$



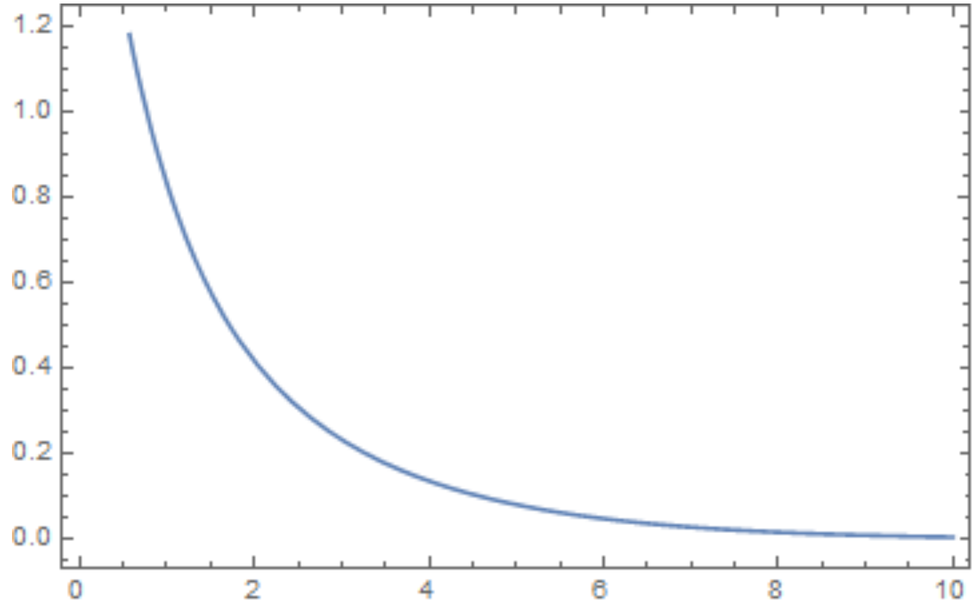
Overdamped: $\beta > \omega_0$

Solution:

$$x(t) = e^{-\beta t}(Ae^{\omega_2 t} + Be^{-\omega_2 t})$$

Where $\omega_2 = \sqrt{\beta^2 - {\omega_0}^2}$

Will not oscillate

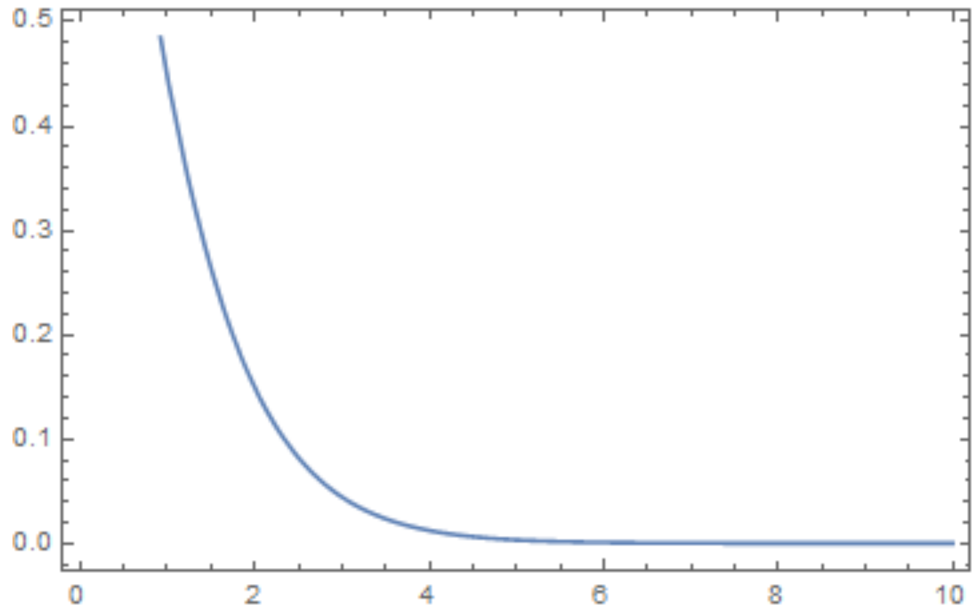


Critically Damped: $\beta = \omega_0$

Solution:

$$x(t) = e^{-\beta t}(A + Bt)$$

Will reach equilibrium in the minimum amount of time.



Damped and Driven Oscillations

Newton's Second Law: $-kx - b\dot{x} + F_0 \cos \omega t = m\ddot{x}$

Equations of Motion: $\ddot{x} + 2\beta\dot{x} + {\omega_0}^2 x = G \cos \omega t$, where $G = \frac{F_0}{m}$

Solutions: $x(t) = x_h(t) + x_p(t)$

$$x_p(t) = A_p \cos(\omega t + \phi_p), \text{ where } A_p = \frac{G}{\sqrt{({\omega_0}^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

Resonance

$$\omega_R = \sqrt{{\omega_0}^2 - 2\beta^2}$$

Quality Factor

$$Q \equiv \frac{\omega_R}{2\beta}$$

Electrical Oscillations

RLC Circuit:

$$\text{Resistor: } V_R = iR$$

$$\text{Capacitor: } V_C = q/C$$

$$\text{Inductor: } V_L = L\frac{di}{dt}$$

EOM:

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = V_0 \cos \omega t$$

$$\updownarrow \qquad \updownarrow \qquad \updownarrow \qquad \updownarrow$$

$$m\ddot{x} + b\dot{x} + kq = F_0 \cos \omega t$$

Properties of Waves

The Wave Equation

$$\frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2}$$

Principle of Superposition:

For any solution to the wave equation of the form *f(x± vt)* and *g(x ± vt),* the function (*f+g)* also solves the wave equation

Example: $f(x,t) = \frac{1}{2}(\cos(x+vt)+\cos(x-vt))$

The Wave Equation

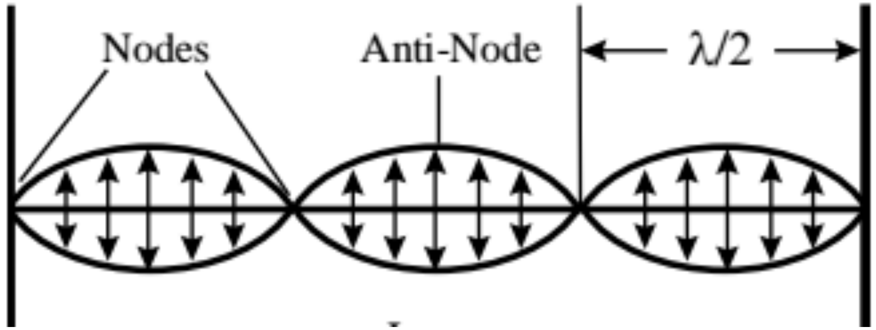
$$\frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2}$$

Principle of Superposition:

For any solution to the wave equation of the form *f(x± vt)* and *g(x ± vt),* the function (*f+g)* also solves the wave equation

Example: *f(x,t)* = $\frac{1}{2}(\cos(x + vt) + \cos(x - vt))$

Also expressed as $f(x,t) = \cos(x)\cos(vt)$, which is more easily recognized as a standing wave.



Basic Equations

$$\boldsymbol{f(x,t) = A\, cos(kx} - \omega t + \ \delta)$$

$$f(x, t) = \mathrm{Re}(A\, e^{i\,(kx-\omega t)})$$

$$A = |A| e^{\,i\delta}$$

A: amplitude
***k:* wavenumber**
***ω:* angular frequency**
***δ:* phase**

$$v = \frac{\omega}{k} \qquad k = \frac{2\pi}{\lambda} \qquad \omega = \frac{2\pi}{T} = 2\pi f$$

White Book Problem (pg 122): What is the absolute value of the relative phase between two waves described by sin(x-vt+π/6) and cos(x-vt)?

Basic Equations

$$\boldsymbol{f(x,t) = A\, cos(kx} - \omega t + \delta)$$

$$f(x, t) = \mathrm{Re}(A\, e^{i\,(kx-\omega t)})$$

$$A = |A| e^{\,i\delta}$$

A: amplitude
***k:* wavenumber**
***ω:* angular frequency**
***δ:* phase**

$$v = \frac{\omega}{k} \qquad k = \frac{2\pi}{\lambda} \qquad \omega = \frac{2\pi}{T} = 2\pi f$$

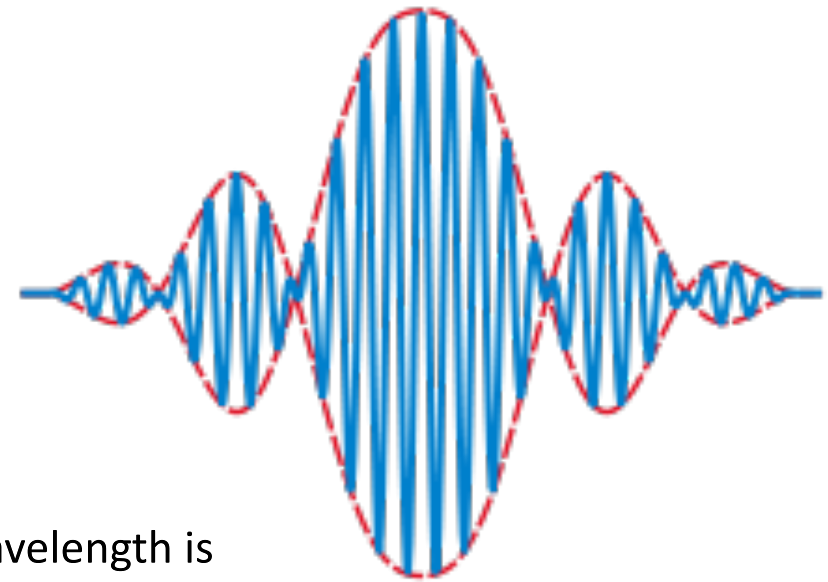
White Book Problem (pg 122): What is the absolute value of the relative phase between two waves described by sin(x-vt+π/6) and cos(x-vt)?

***Answer**: Inherent phase shift between sin and cos is $\frac{\pi}{2}$. Cos is shifted left of sin, so π/2- π/6= π/3*

Phase and Group Velocity

$$v_p = \frac{\omega}{k}$$

$$v_g = \frac{d\omega}{dk}$$



$$\frac{\omega}{k} = \frac{c}{n}$$ (True for EM waves. Note: the wavelength is modified, the frequency stays the same)

Red dotted line shows the envelope of wave packet that travels at the group velocity. Individual crests travel at the phase velocity, which can be a function of k if ω(k). Therefore different wavelengths can have different speeds (dispersion).

Poynting Vector

Using Maxwell's equations, we can find the solutions:

$$\mathbf{E}(\mathbf{r}) = E_o\, e^{i\,(\boldsymbol{k}\cdot\mathbf{r}\,-\omega t)}\,\mathbf{n}$$

$$\mathbf{B}(\mathbf{r}) = \frac{1}{c}\,E_o\, e^{i\,(\boldsymbol{k}\cdot\mathbf{r}\,-\omega t)}(\mathbf{k}\times\mathbf{n})$$

k is the direction of propagation and **n** is the direction of polarization (only for **E**).

Poynting Vector describes the transport of energy

$$\mathbf{S} = \frac{1}{\mu_0}\,(\mathbf{E}\times\mathbf{B})$$

$$I = \langle S\rangle = \frac{1}{2}\,c\epsilon_0\,E_0^{\,2}$$

I is the *intensity, or the average* power per unit area

Malus’s Law

$$\mathbf{E}(x,t) = \mathrm{E}_0\, cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \ \delta)\mathbf{n}$$

$$I\ = I_0 cos^2\theta$$

$$I\ = \frac{1}{2} I_0$$

$$I\ = \frac{1}{2} I_0$$

$$I\ = 0$$

Malus's Law

$$\mathbf{E}(x,t) = \mathrm{E}_0\, cos(\mathbf{k}\cdot\mathbf{r} - \omega t +\ \delta)\mathbf{n}$$

$$I = I_0 cos^2\theta$$

White Book Problem (pg 121): Polarized light with polarization vector **n**=2**x**+3**y** is incident on a polarizer oriented at **v**=**x**+2**y**. What is the ratio of the intensity of transmitted light to the initial intensity

Malus's Law

$$\mathbf{E}(x,t) = \mathrm{E}_0\, cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \ \delta)\mathbf{n}$$

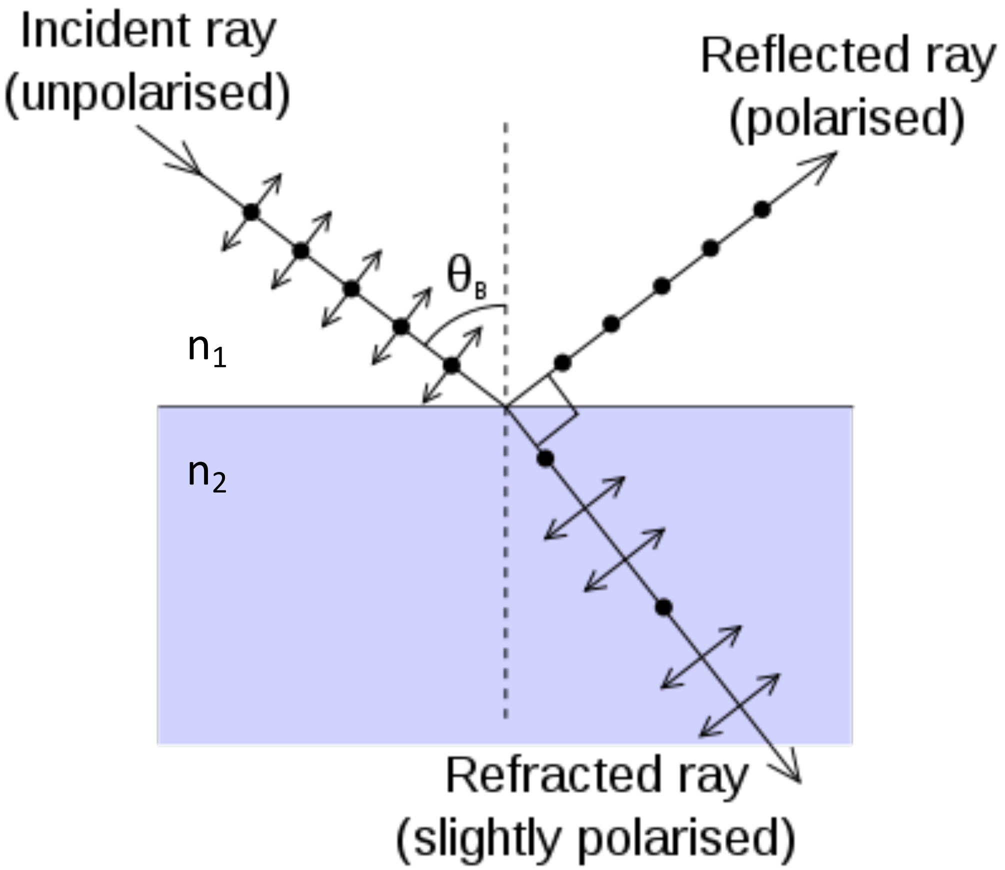
$$I = I_0 cos^2\theta$$

White Book Problem (pg 121): Polarized light with polarization vector **n**=2**x**+3**y** is incident on a polarizer oriented at **v**=**x**+2**y**. What is the ratio of the intensity of transmitted light to the initial intensity

$$\frac{I_t}{I_0} = \frac{\boldsymbol{n}\cdot\boldsymbol{v}}{(|n||v|)^2} = \left(\frac{(2+6)}{\sqrt{13}\sqrt{5}}\right)^2 = \frac{64}{65}$$

Brewster's Angle

$$\theta_B = \tan^{-1} \frac{n_2}{n_1}$$



Interference and Diffraction

Interference

$$f(x,t) = A\cos(kx - \omega t)$$
$$f(x,t) = A\cos(kx - \omega t + \delta)$$

Type of Interference	Phase Shift (δ)	Examples
Constructive	-Odd multiple of π - **2mπ m=0,1,2…**	- Double-slit interference - Single-slit diffraction
Destructive	-Even multiple of π - **(2m+1) π m=0,1,2…**	- Thin films

Most interference problems on the GRE rely strictly on memory.

Double-Slit Interference

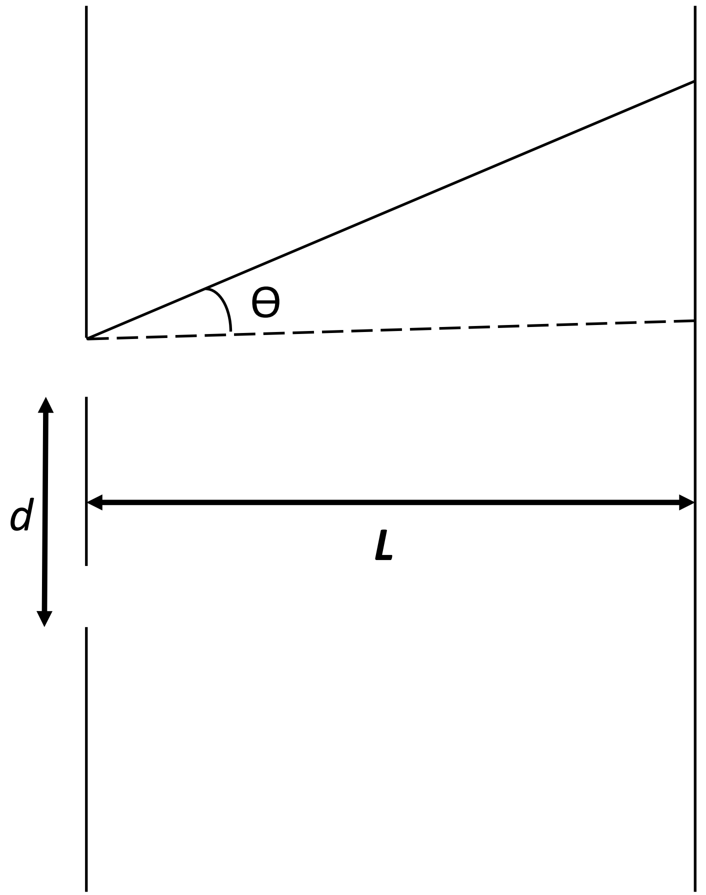
Monochromatic point source (λ)

Maxima

$$d sin\theta = m\lambda$$

Minima

$$d sin\theta = (m + \frac{1}{2})\lambda$$



Single-Slit Interference

Monochromatic point source (λ)

Assume *L>>d*

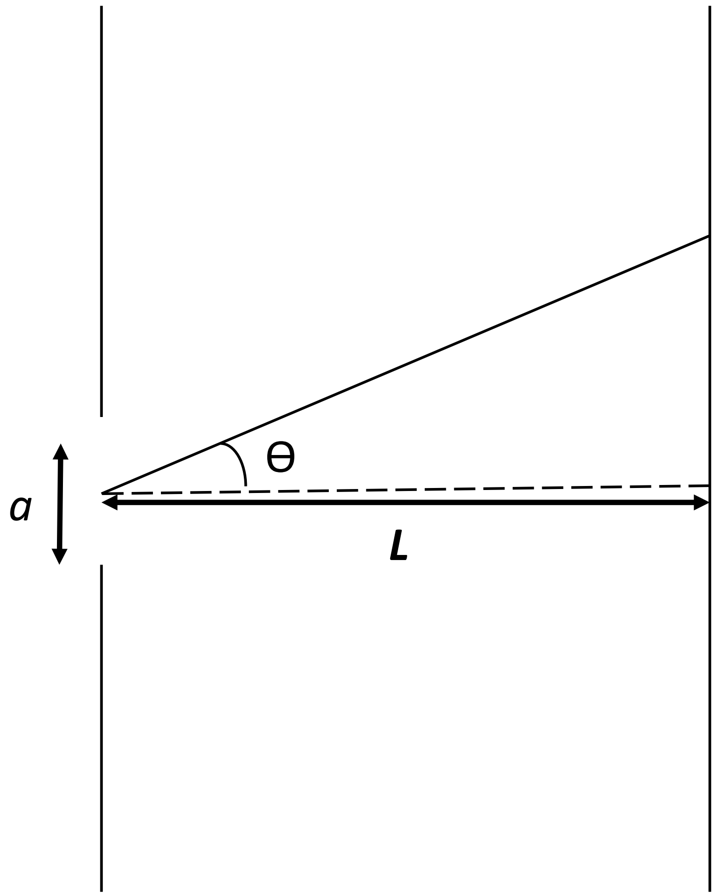
Minima

$$a sin\theta = m\lambda$$

Maxima

no simple formula

central max at center

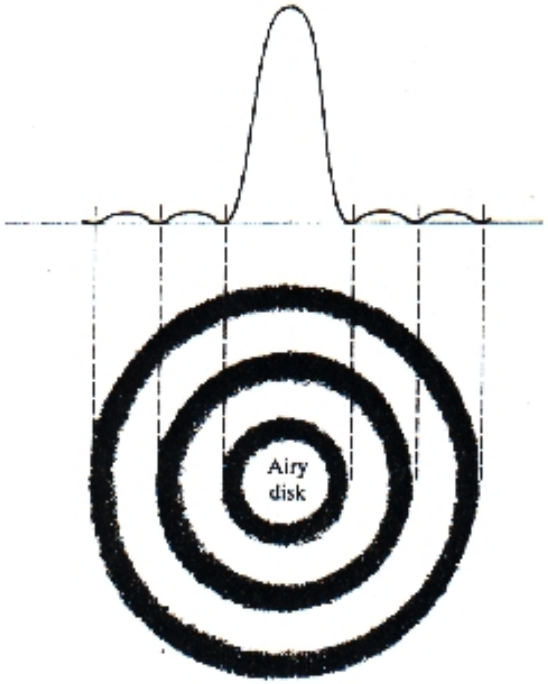


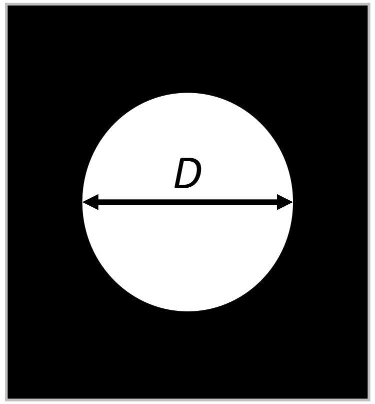
Diffraction by a Circular Aperture

GRE only tests limiting case of the first diffraction minima.

First Circular Diffraction Minima

$$D sin\theta = 1.22\lambda$$





(Left) The resultant diffraction pattern of a circular aperture.
(Right) A circular aperture.

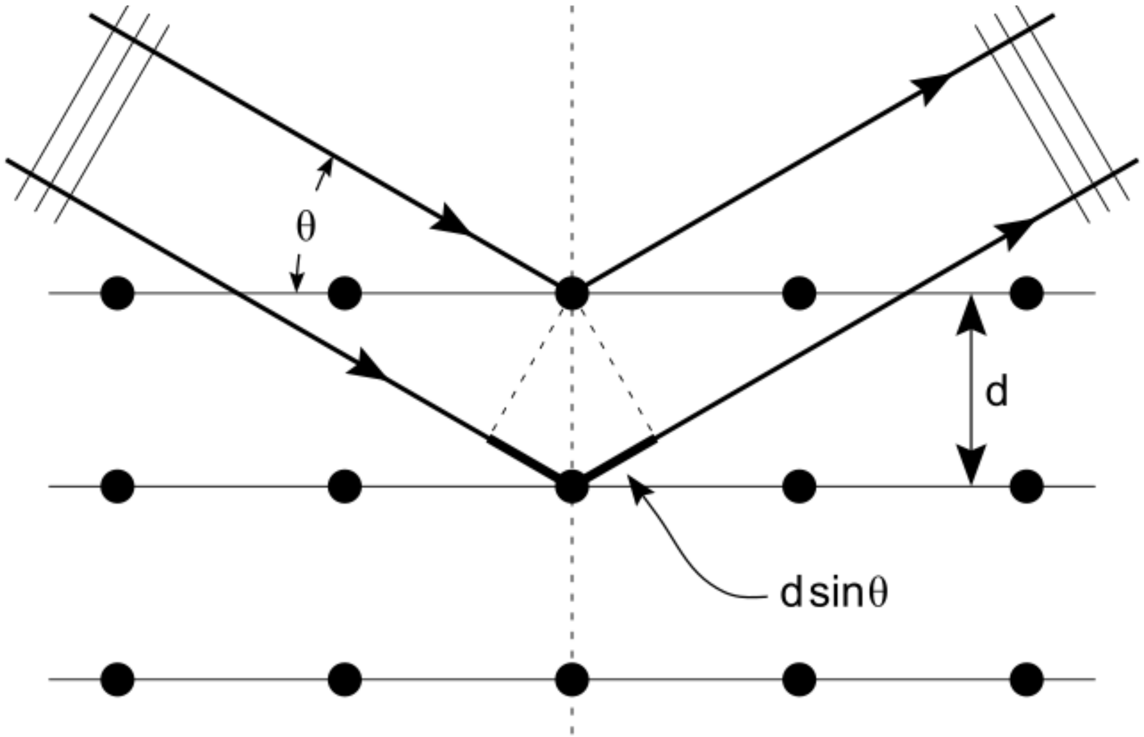
Bragg Diffraction

X-rays incident on a crystal lattice.

Like double-slit, interfere by pld.

Maxima

$$d \sin\theta = n\lambda/2$$

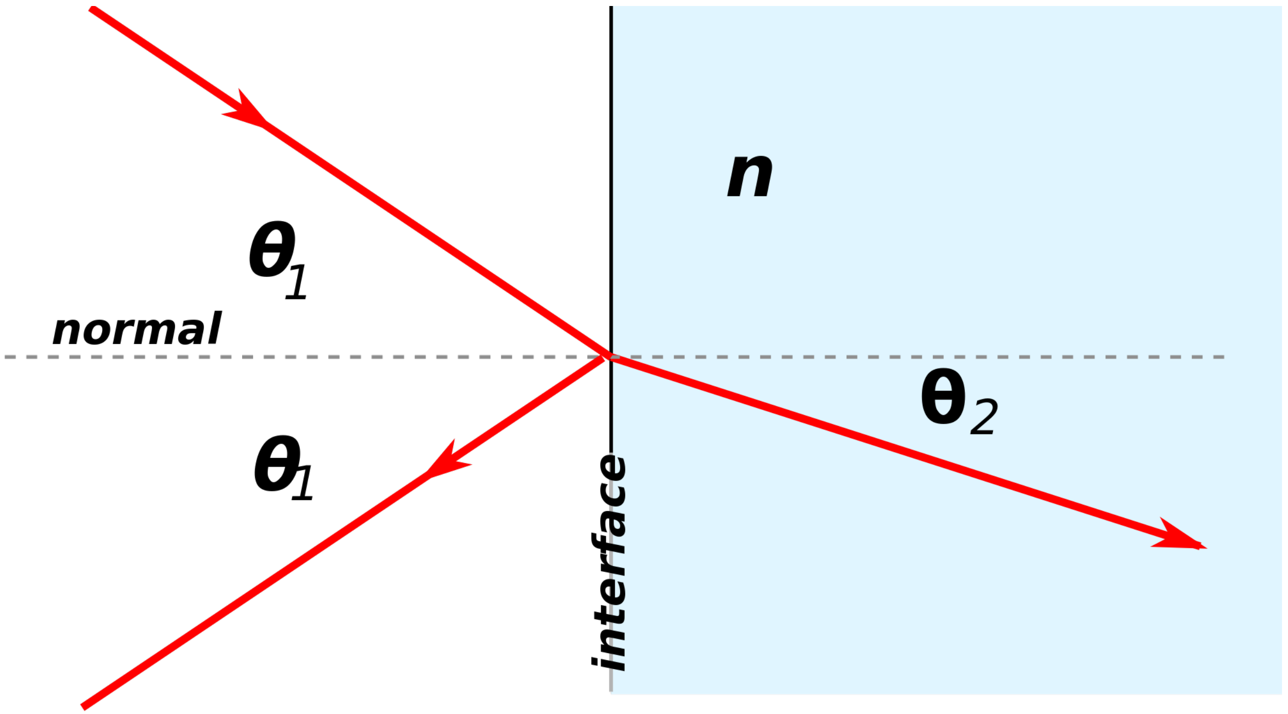


Geometric Optics

Assorted Topics

Reflection & Refraction

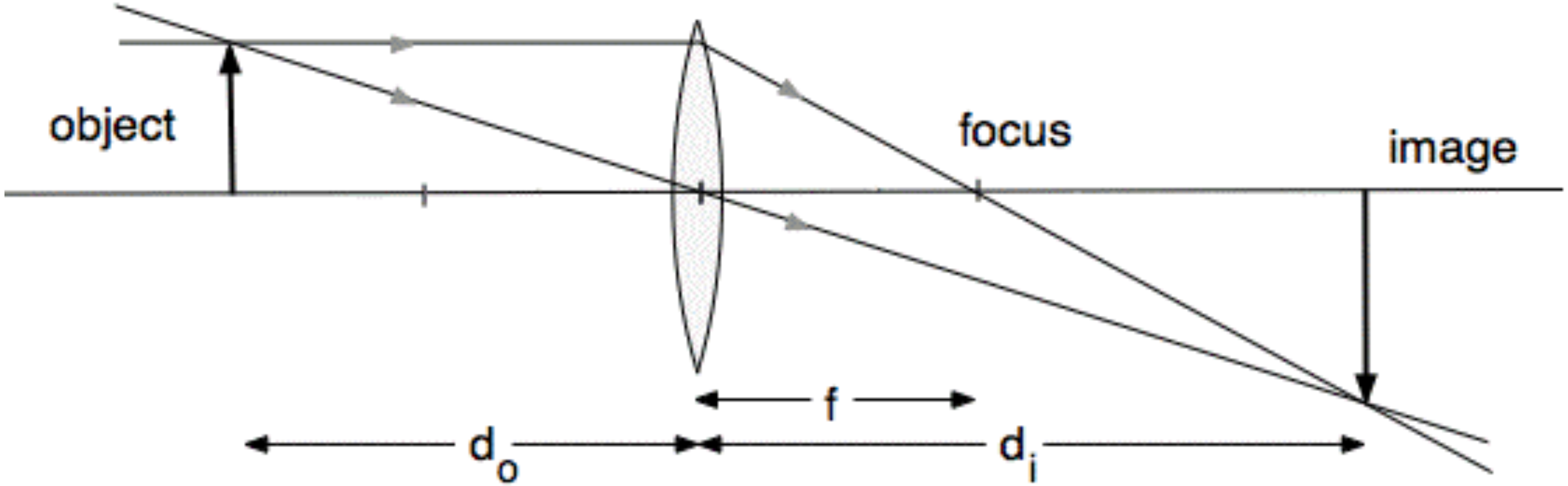
$$\theta_i = \theta_r$$



$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

n = index of refraction

Lens Equation



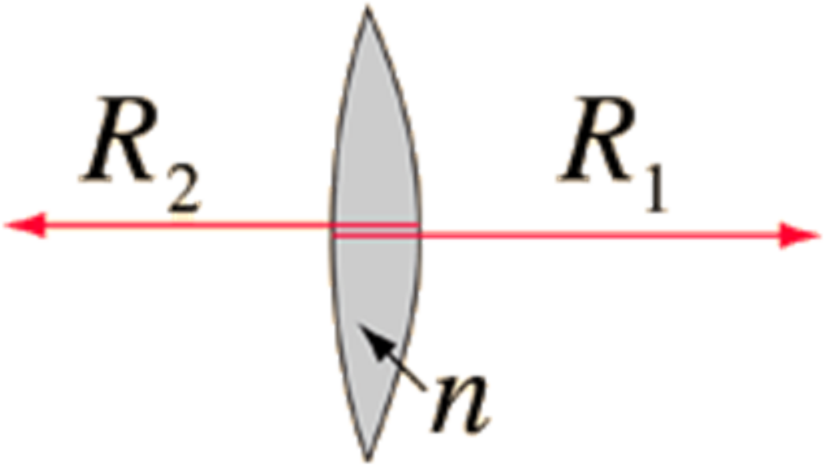
d_o = object distance

d_i = image distance

f = focal length

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$$

Lensmaker's Equation



$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

f = focal length

R = radius of curvature

n = index of refraction

$R_1 = +$

$R_2 = -$

Rayleigh Scattering

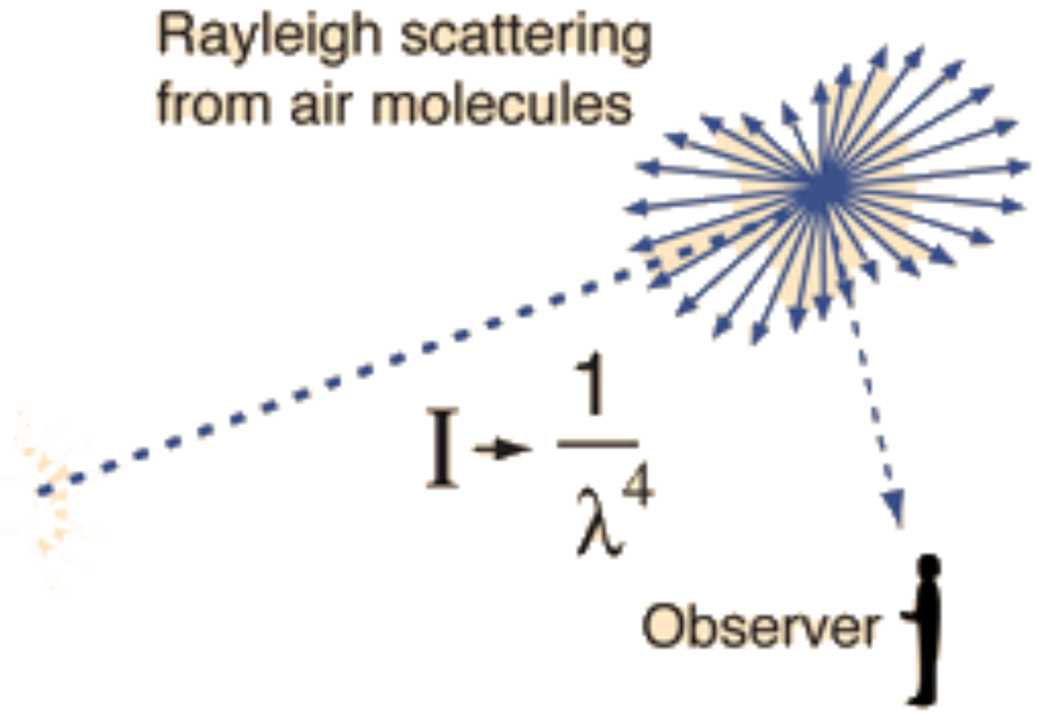
$$\text{for } \lambda \gg a$$

$$I \propto I_0 \lambda^{-4} a^6$$

$$I = \text{intensity}$$

$$\lambda = \text{wavelength}$$

$$a = \text{particle size}$$



Doppler Effect

$$f = f_0\left(\frac{v+v_r}{v-v_s}\right)$$

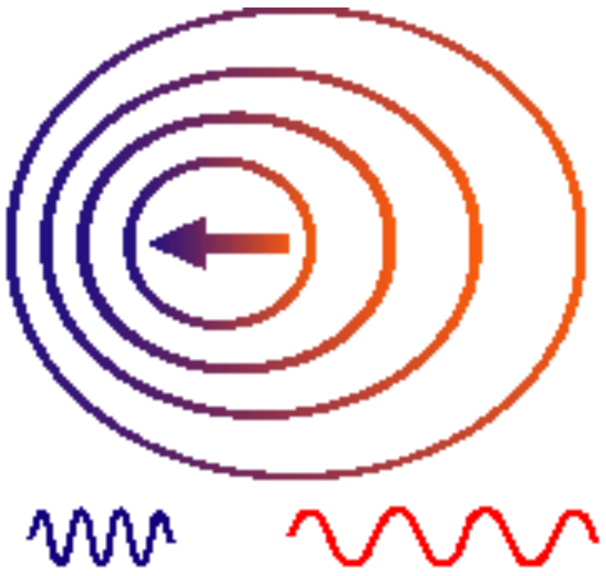
f = observed frequency

f_0 = emitted frequency

v = wave velocity

v_r = receiver velocity

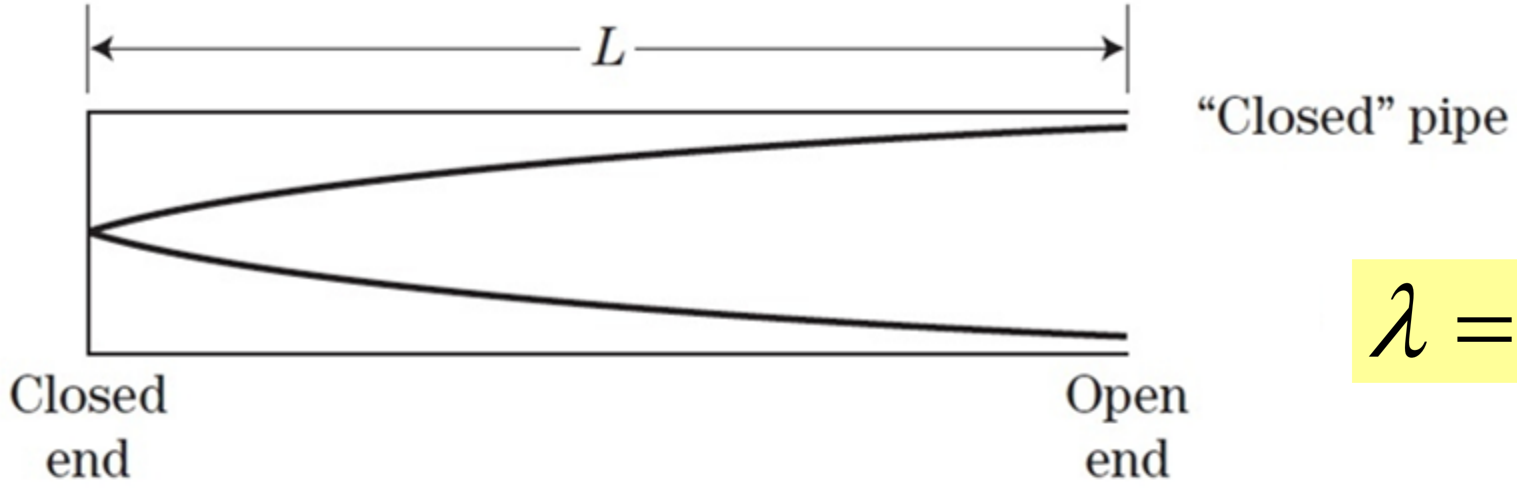
v_s = source velocity



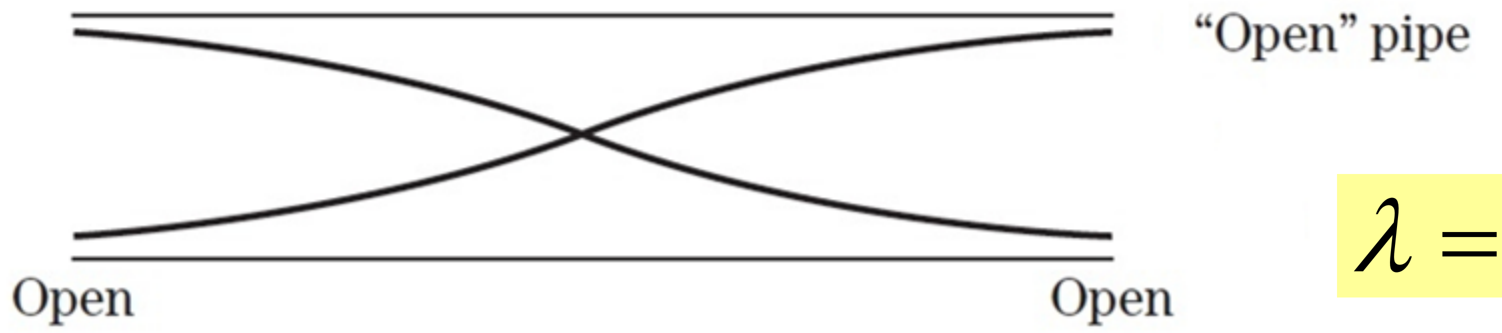
$v_r \,\&\, v_s = +,\ \mathrm{s} \rightarrow\leftarrow \mathrm{r}$

$v_r \,\&\, v_s = -,\ \mathrm{s} \leftarrow\rightarrow \mathrm{r}$

Standing Sound Waves



$$\lambda = 4L$$



$$\lambda = 2L$$