Part C: Vibration & Oscillation

Chapter 10. Repeating Motion

10a. Period & Frequency

Although vibrations are not themselves sound (in the sense of the objective entity that travels through air), the two are intimately connected. Most sounds can be traced back to some object vibrating. In many situations, sounds cause an object to vibrate. Indeed, when we discuss perception of sound, we really mean perception of a vibration in the ear. And later chapters will describe how the air itself is vibrating as sound passes through it.

Sound travels across distances, and also involves changes with time. That makes the vibration of air in sound more complicated than the vibration of objects, which do not travel. So, we will first discuss objects. The words **vibration** and **oscillation** both refer to back-and-forth motion. Usually, “vibration” is used for motion that is quite fast, while “oscillation” is used for slower motion. But their characteristics are the same, and the words will be used interchangeably in this book.

To have a vibration, there must be an object that vibrates, and there must be something causing it to move. If the cause is complicated, or even sentient (like a person shaking a tambourine), then the possible motions could be almost anything. In the physics spirit, in this Part we’ll focus on relatively simple causes for the motion. The cause of motion and the object are considered together as a **system,** which can have properties that neither of the components have on their own. The way the system moves (without any outside influence) is called its **natural motion.**

In general, in order for a system to oscillate on its own, you need two conditions. First, there must be a special position for the object (or possibly a range of positions) where the cause of motion has no effect, so that the object could remain stationary there indefinitely. This is the **equilibrium position** for the system. Second, if the object ever moves from the equilibrium position, there must be something that pushes the object back towards equilibrium, called the **restoring force.** The concept of force is detailed starting in Chapter 15.

A key feature of simple vibrations is that their motion repeats. Each repeating unit is called a **cycle.** That is, a cycle is the shortest set of motions which, if replayed over and over, would describe the vibration for as long as it lasts. Notice that a cycle is not something that can be measured with a single number. Also note that for a given vibration, you can choose to start a cycle at any point. Suppose a vibration is represented by the repeating pattern …ABCDEABCDEABCDE…, where each letter might represent a small motion that is part of the vibration. Then ABCDE would be a cycle, but so would CDEAB and EABCD.

The time required for one cycle is called the **period** of the oscillation, and it is given the particular symbol T. In conversational English, “period” can mean any time interval of interest, but in physics “period” has a much more specific meaning. Anything that is repeating can also be called **periodic,** meaning that is has a period.

You might notice that we have already used T for temperature in Eq. 7.1. We try to avoid having the same default variable for different concepts, but in the end there are just too many concepts and too few letters. While working a physics question, you certainly *must* have a different variable for every individual measurement. If you were working a question involving both temperature and period, that would be a good time to distinguish them by subscripts.

Since it is a particular interval of time, the root unit for period is the second. For audible sounds, milliseconds are typically handy. However, it is often helpful to consider period to be measured in the unit seconds/cycle. A cycle is what physicists call a **dimensionless unit.** This book will not address the details, but the result is that "cycle" can be inserted or removed from a derived unit whenever it makes sense. It functions outside the rules about units combining algebraically. As a result, both s and s/cycle are valid units for period.

If you want to observe some number of cycles N, the elapsed time t required is proportional to N. Therfore, for any particular vibrating motion, the ratio of time interval to number of cycles is always the same. A particularly useful corresponding pair is that 1 cycle lasts for one period, giving

$$\frac{t}{N} = \frac{T}{1} = T \quad . \tag{10.1}$$

There is another often-used parameter that is sort of the alter ego of the period, in that it conveys the same information about the motion. Especially when dealing with very rapid motion, rather than how much time each cycle takes, it can be easier to consider how many cycles are completed in each second. This is called the **frequency** of the oscillation, denoted by *f*. The root unit of frequency is cycles/second, which hints at the mathematical relationship between frequency and period,

$$f = \frac{1}{T} \quad . \tag{10.2}$$

Because it is used so much, cycles/second is given the name hertz, abbreviated Hz, named after a student of the sound scientist Helmholtz. More about Helmholtz is in Chapter 28.

Often when considering a vibration, we imagine that it goes on forever, with no beginning or end. But any real vibration does have a beginning and end, and the time interval during which the motion occurs is called the **duration**. Duration doesn't have any special algebraic symbol. Take particular notice of the difference between the duration and the period of an oscillation.

10b. Displacement

Before getting into how systems oscillate, we need more tools for describing motion. Our definition of speed in Eq. 4.21 relied on the fact that sound always travels at a steady speed, but oscillating objects keep changing speed. We need to distinguish between a few kinds of speed, which first requires distinguishing between a few kinds of distance.

Consider the motion of the tip of the tine of a tuning fork (see Figure 10.1). Suppose that the extremes of motion are 2 mm apart and that the motion repeats itself every $T = 3.8$ ms (which makes this a tuning fork for the pitch "middle C"). In particular, consider the cycle during which the tine moves from the far left to the far right, and then back to the far left.

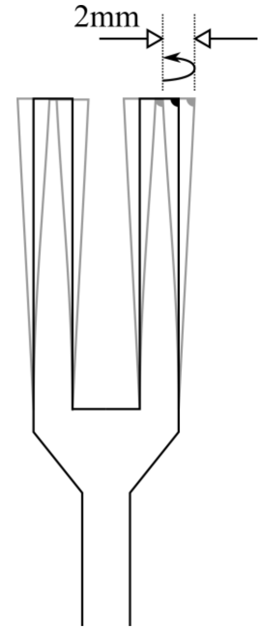

Figure 10.1

Tuning fork vibrating. Arrows relate to the motion of the dot on the upper right corner.

Distance

There is a sense in which the tine has moved 4 mm during that time. This is what physicists call distance traveled, or just **distance**. Distance is always a positive number. Often *d* is chosen as the algebraic symbol.

Displacement

If you compare the beginning and ending positions, without regard for where the tine was in between, there is another sense in which the tine has not moved at all. Physicists call this idea **displacement,** and in this case the displacement is zero. When an object does not end in the same place as it started, completely describing the result (but not the full motion) requires giving both the straight-line distance between the beginning and ending positions, *and* the direction of that line. It turns out to be handy to lump both distance and direction together, and to call the combination the **displacement.** The fact that direction is included makes displacement a **vector**.

In general, to describe a direction with a number, you need to first specify a special reference direction, for instance "towards North" or "to the right on the page." Then any other direction can be specified by angles away from the reference direction.

But luckily, for oscillations there are only two directions involved, because the motion is along a single axis. That is, the motion is **one-dimensional**. If we choose to call one direction positive and the other negative, then the vector idea reduces to simply a **signed number**. In principle, either direction can be chosen as positive. But nearly always, positive is chosen to be rightward or upward.

The **magnitude** of a vector is its size. That is, it is all the information in the vector except its direction. So, we could correctly say, "For any motion, the **magnitude** of the **displacement** due to that motion is the straight-line **distance** from the beginning point to the ending point." For our one-dimensional case, magnitude is obtained just by taking the **absolute value.**

It is not necessary to have an actual object moving in order to refer to a displacement. The displacement from point A to point B is still just a distance and direction, imagining the points to be the start and end points of a hypothetical motion.

Position

If we want to quantify where an object is, we can do so by first choosing a special point of reference, called an **origin**, and choosing a reference direction. We can then specify a **position** as a displacement from that origin. Since a position is a special kind of displacement, it is a vector. A common choice for the algebraic symbol of position is r. But again, for oscillations, choosing a reference direction reduces to choosing which direction along the axis of motion to call positive. Positions are then just signed numbers. In that situation, the usual variable name choice is x, or sometimes y or z.

Once an origin has been chosen, the displacement between two other places is given by the difference of their positions. For this reason, a common algebraic symbol for displacement is Δx. The relation to position is then given by

$$\Delta x = x_{\text{end}} - x_{\text{begin}} \quad . \tag{10.3}$$

Notice that this has the pattern "later value minus earlier value," or "final minus initial" or "ending minus beginning." This pattern usually requires a little attention because the parts are reversed from the order in which they occurred.

Very likely, you once learned in a math class to use the variable x to represent "the unknown" in a question. In physics, that is *never* done. Algebra is already abstract enough; it's a bad plan to make the equations even more obscure by using a generic x. In principle, algebra allows you to use any symbol that you desire to represent any quantity. But in practice, it is strongly recommended to use variables that remind you, as well as anyone reviewing your work, of what type of physical quantity it represents. Reserve x for position.

Chapter 11. Position Graphs

In order to describe motions like oscillations, in which positions and speeds are continually changing, we need more than a single number. One way to describe the continual changes is with **graphs**, pictorial representations of the relationship between two variables.

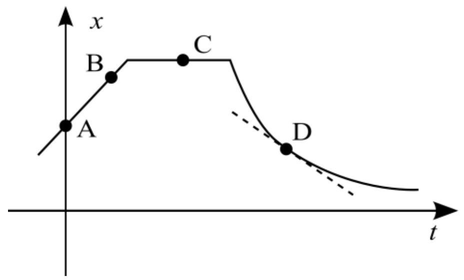

Figure 11.1
Graph of position versus time.

To show how things change, we need to use one axis to represent time, as in Figure 11.1. Nearly always, the horizontal axis is chosen for time. A time axis is more abstract than a position axis, as we are representing intervals of time with horizontal displacements on paper. However, the same basic principles apply as for any graph, including that there is a horizontal scale factor, which now might have units like seconds per centimeter (s/cm). That scale might be indicated by tick marks along the axis, but Figure 11.1 has been left without them because it will be used to illustrate ideas that hold for any scale.

Only the vertical axis remains to represent position information. But since our goal is to describe one-dimensional oscillation, position along one line will be sufficient for our purposes. The result is a position versus time graph (often shortened to position-time graph). Notice that the labels for the axes are not x and y, as they would be for position axes; the labels change to reflect the quantity they represent. Since the two axis scales are different (indeed, they have different units), the length of a diagonal line on the graph has no useful meaning.

Chapter 12. Velocity

If we take the idea of speed, and replace the distance in Eq. 5.2 with displacement, we get a quantity that physicists call **velocity,**

$$v = \frac{\Delta x}{\Delta t} \quad . \tag{12.1}$$

Only lowercase v is ever used to represent velocity. Its units are, not surprisingly, the same as for speed. Even though the denominator represents the same sort of elapsed time as in Eq. 5.2, it is represented with Δ to emphasize the parallel with the displacement.

Velocity includes the direction that it inherits from the displacement Δx. For this book, that just means that velocity can be positive or negative, indicating whether the object's motion is towards or against the chosen positive direction. If something moves with constant speed and always in the same direction, as sound often does, that is called **uniform motion.** In that case, the magnitude of the velocity equals the speed.

Average Velocity

But for oscillations, the speed, and even the direction, of motion are *not* constant. Similar to how average speed was defined, Eq. 12.1 can still be applied to such inconstant motion, with a result called **average velocity.** As with average speed, the average velocity depends not only on the motion itself, but also on how the initial and final times are chosen. And again, the calculation will result in a velocity that is roughly in the middle of the velocities that occur between the chosen moments, but this sort of average is not simply related to an arithmetic mean.

For instance, consider the average velocity of the tuning fork tine in Figure 10.1 as it moves from extreme right to extreme left. That's one half of a cycle, taking a time $\Delta t = T/2 = 1.9$ ms. Making the usual choice that rightward is positive, the average velocity is

$$v_1 = \frac{-2\ \text{mm}}{1.9\ \text{ms}} = -1.053\frac{\text{m}}{\text{s}} \quad . \tag{12.2}$$

For some time intervals, the magnitude of the average velocity will be different from the average speed. Consider the tine's average velocity calculated over a single cycle. No matter how we choose the start of the cycle, the tine's position at the end of the cycle must be the same as at the beginning, so that the motion is ready to start into the next cycle. Thus $\Delta x = 0\ \mathrm{m}$ and $v_2 = 0\ \mathrm{m/s}$ exactly. Compare that to the tine's average speed over one cycle

$$s_2 = \frac{4\ \mathrm{mm}}{3.8\ \mathrm{ms}} = 1.053\frac{\mathrm{m}}{\mathrm{s}} \quad . \tag{12.3}$$

This occurs as a result of the change in direction, which results in a distance traveled that is different from the magnitude of the displacement. So in general, the average speed will equal the magnitude of the average velocity only when there is no change in direction between the start and finish.

We might calculate the average velocity over a very large number of cycles. Although the tine travels over a greater and greater distance as time goes on, the displacement can never be larger than 2 mm. So as the Δt in the denominator gets larger, the average velocity must get very small. The longer the time interval, the closer the average velocity is to zero. This result is not unreasonable. Certainly, there is a sense in which the vibrating tuning fork never actually travels anywhere. And depending on the choice of initial and final positions, the velocity can be positive or negative, so that an average of zero is not a surprise mathematically. But these observations don't help us understand the details of that motion. For that, we need the next few chapters.

Chapter 13. Graphs and Velocity

Position versus time graphs are useful for more than simply illustrating how an object moves in time. For instance, consider the motion in Figure 11.1 from point A to point B. Because the graph is a straight line between them, for any times between A and B the distance traveled is proportional to the time of travel. That is, during that time the motion has a constant, uniform velocity (and speed). To calculate the velocity, we have

$$v = \frac{\Delta x_{AB}}{\Delta t_{AB}} = \frac{x_B - x_A}{t_B - t_A} = \frac{\text{change on vertical axis}}{\text{change on horizontal axis}} \quad , \tag{13.1}$$

which is the definition of the **slope** of a graph.

Slope on a position versus time graph gives the velocity of the motion.

This idea is useful even when velocity is not constant. For the motion from point A to point C, with two parts having different velocities, the same considerations show us that the *average* velocity between A and C is given by the slope of a straight line between those points, even though that straight line does not describe the details of the motion.

Instantaneous Velocity

In fact, this slope-velocity relationship provides us with a new way to handle non-constant velocity, one that would have been quite difficult to use without the graph. Consider the motion near point D, where the curving line tells us that the velocity is not constant for *any* time interval. We could calculate a variety of average velocities by choosing different beginning and ending points near D. However, there is only one **tangent line** that matches the curve's slope exactly at point D. It is the dashed line in Figure 11.1. The slope of that line is the **instantaneous velocity** at point D. The instantaneous velocity is a bit harder to calculate from a graph than an average velocity, since it requires determining the tangent line. But conceptually it has a big advantage over average velocity in that it is specific to a single moment in time, instead of depending on both a beginning and ending time.

Notice that when the velocity is constant, the tangent line at a point (for instance, at point B in Figure 11.1) passes exactly along the actual graph line. Thus, the instantaneous velocity and the constant velocity match perfectly.

The magnitude of the instantaneous velocity is the **instantaneous speed.** This brings us back to the situation where speed equals the magnitude of the velocity. For ease of use, the word "instantaneous" is often dropped from these names. However, the word "average" is *never* dropped from the terms average velocity or average speed.

Chapter 14. Acceleration

Position, displacement, and velocity are all elements of the subject of **kinematics**, the description of motion. In all sciences, accurately describing the subjects of study is an important first step. However, sciences then make progress by finding organizing principles and, if possible, instances of cause and effect. Therefore, the question arises, "What *causes* motions, in a general sense?" Answering this question is the subject of **dynamics**. Everyday experience provides a ready answer: motions are caused by pushing, pulling, or more generally applying a **force**. A later chapter will consider quantifying force; for now, the everyday concept will suffice.

Do forces directly cause objects to have velocity? For a very long era in Western thought, that was the prevalent model, attributed to Aristotle. It was thought that as long as an object is moving, it must be experiencing a force. However, this model has difficulty describing many situations. For instance, if you throw a ball, it continues to move long after your hand has stopped pushing it. And if this book starts to slide off your lap, you could stop it by applying a force, which would be an example of a force *removing* velocity. Aristotle's model needed extra rules to account for these things.

A new model was eventually found, of which Galileo was the most famous proponent. This model said that forces cause *changes in* velocity, and ultimately it explains a much wider range of situations with many fewer assumptions. In fact, this model has been extraordinarily successful, accurately predicting results in all physical situations except on the very smallest scales, where quantum mechanics is required.

Being successful, or at least useful, is necessary for a model to gain credibility in physics. But the fact that it has fewer assumptions is also important—physicists would call the model "elegant." In fact, as physics has advanced, it has generally turned out that the more elegant a model, the more likely it is to be successful. In this respect, it seems that nature tends to be simple. This doesn't necessarily mean easier to understand. Changes in velocity are more abstract than velocity itself, and therefore harder to think about. But it does mean that there are fewer underlying rules.

In order to use this model for dynamics, we must first expand our knowledge of kinematics to include a way to describe changes in velocity. We have already seen how velocity is a useful description of the rate at which position changes. So, it is a natural extension to use the same mathematical structure to describe the rate at which velocity changes, which physicists name **acceleration**. For instance, if the velocity of a car changes during an interval of time, we can define its **average acceleration** as

$$a = \frac{\Delta v}{\Delta t} = \frac{v_{\mathrm{end}} - v_{\mathrm{start}}}{t_{\mathrm{end}} - t_{\mathrm{start}}} \quad , \tag{14.1}$$

Since velocity is a vector idea, this equation says that acceleration is also. But again, for our purposes of describing motion along a single line, this simply means that acceleration is a signed quantity. Only lowercase a is ever used for the algebraic variable. As always, the defining equation tells us what units can be used for acceleration; the example with SI root units is $(\mathrm{m/s})/\mathrm{s} = \mathrm{m/s^2}$.

Acceleration is an idea that we all have direct experience with, for instance while being in a vehicle that changes its velocity over a relatively short interval of time. And we all know that the experience of acceleration is very different from the experience of traveling down a highway at constant speed (high velocity, but zero acceleration). Nevertheless, there are many situations where it is easy to confuse these two aspects of motion.

One such situation is in determining the direction of acceleration. One approach is to carefully use Eq. 14.1 to determine the sign of a. This can be done as long as you know the signs and relative sizes of the velocities; actual numbers are not necessary. For example, suppose that you throw a ball up into the air and consider a time interval from just after the ball leaves your hand to just before the ball reaches its highest point. Taking upwards to be positive, $v_{\text{begin}} > v_{\text{end}} > 0$ (ball is moving in positive direction but slowing down). This implies that $v_{\text{end}} - v_{\text{begin}} < 0$, so the acceleration is in the negative direction.

Another approach which may be useful is to anthropomorphize the moving object, and ask, "In which direction is it *trying* to move?" For instance, even though the thrown ball is moving upwards, we can imagine that it is "trying" to go down, again giving a negative acceleration.

One of the original motivations for defining acceleration was the following very useful model, first studied thoroughly by Galileo Galilei.[5]

> Any object that has a reasonably large weight, and is reasonably compact in shape, when near the surface of the earth and not in contact with anything else, moves with an acceleration of $9.8\,\mathrm{m/s^2}$ in the downwards direction.

Such an object is said to be in **free-fall**. This special value is named **the acceleration due to gravity**, denoted by g. The next digit, the $0.01\,\mathrm{m/s^2}$ place, depends on where you are on the earth, but it is quite close to 0, so that using just two significant figures gives you better than 1% accuracy. The usual custom is to *not* treat g as a vector quantity, but as a positive numerical quantity. Thus, if we choose the usual axis for vertical motion, with the upward direction being positive, the above principle is expressed by the equation

$$a_{\text{free-fall}} = -g \quad . \tag{14.2}$$

Be careful to use this value only in appropriate cases. If there is *anything* other than gravity influencing an object's motion, then its acceleration is highly unlikely to be g. Inappropriately assuming that an acceleration is $9.8\,\mathrm{m/s^2}$ is a common error in introductory physics courses.

In a typical introductory physics course, because of the constant acceleration due to gravity, the kinematics of constantly accelerated motion (also called projectile motion) is studied extensively. However, our goal is to describe oscillations, for which even the acceleration is not constant. Gravity will be useful in a few parts of this book. But to make progress, we need to move directly to the cause of acceleration: force and the subject of dynamics.

Chapter 15. Force and Acceleration

Force is a pretty familiar, everyday idea. But we need to be able to quantify forces, so that we can write equations about them. One element we can notice right away is that forces push or pull in particular directions: they are vector quantities. For the purposes of describing motion along a line this means, once again, that we can expect forces to be signed quantities.

In the early 1600s, two important English scientists invented two different ways to quantify forces.

Isaac Newton chose the relationship between force and acceleration as his basis. Since what we would intuitively describe as larger forces result in larger accelerations, Newton proposed that force be *defined* as being proportional to the resulting acceleration, all other things being equal. This is expressed by

$$F_{\text{on}} \propto a \quad , \tag{15.1}$$

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[5] Galileo Galilei, *Dialogs Concerning Two New Sciences,* trans. Henry Crew and Alfonso de Salvio (New York: Dover Publications Inc., (1638) 1954).

where F is the standard symbol for force, and the subscript “on” reminds us that the force is applied *on* the object.

While one can always choose to make this definition for accelerating a single object, it is conceivable that it might fail to be consistent if you apply the same forces to different objects. If force A accelerates a car twice as much as force B, does that guarantee that force A also accelerates a ball twice as much as force B? Is force A twice B for any object at all? Centuries of experience give us the answer, “Yes.” This model has been spectacularly successful.

Of course, any given force will give the ball a much larger acceleration that it will give the car; cars are much harder to move. As usual with proportional quantities, it is useful to take their ratio. In this case, the ratio of force applied on an object to the resulting acceleration describes how hard it is to accelerate that object, and is defined as the object’s **inertial mass,** usually shortened to just **mass** and denoted by m. Rewriting that ratio gives one of the most famous equations of physics, **Newton’s Second Law**

$$\frac{F}{a} = m \quad \Rightarrow \quad F = ma \quad . \tag{15.2}$$

Although it arises as a sort of side-effect in Eq. 15.2, mass is a fundamental type of quantity in the SI system, like distance and time. More intuitively, it describes how much material, or how much “stuff,” an object has. Linking directly to Eq. 15.2, mass describes how hard it is to shake the object back and forth. There is also an extremely close connection to how hard it is to lift the object, which is explored in Section 18b.

These equations require some new units to be defined. The SI root unit of mass is the gram, with the symbol g. Other units of mass can be obtained by adding prefixes to gram in the usual way. However, the SI base unit of mass is the kilogram (kg), which means that the kilogram is used to build named derived units. This is the only place in SI where the base and root units are different, which can be a source of difficulty. Usually, the safest method is to always use kilograms in calculations. Something with a mass of 1 kg weighs a little more than two pounds.

Equation 15.2 tells us that the SI basic unit for force must be $(\mathrm{kg} \cdot \mathrm{m/s^2})$. Because this is such a mouthful, this derived unit is given a name of its own, the root unit newton (N). A newton of force is roughly equal to a quarter pound. Keep in mind that the newton has the kilogram (not gram) inside. When working with small objects, with masses given in grams, you must remember to convert to kilograms before combining with newtons.

Notice that when a force pushes in one direction, the pushed object always accelerates in that direction (even when its velocity is in the opposite direction, such as when the object is being slowed to a stop). Therefore, F and a will either both be positive or both be negative, making m always positive in Eq. 15.2.

Chapter 16. Force and Springs

If a force is applied to a spring (or anything springy), then it deforms, either stretching or compressing or deflecting. Larger forces result in larger deformations. In the early 1600s, Robert Hooke chose this relationship as his basis for quantifying force. Hooke proposed that force be *defined* as being proportional to the resulting deformation, all other things being equal. This is expressed by

$$F_{\mathrm{on}} \propto \Delta L \quad , \tag{16.1}$$

where F_{on} is the force *on* the spring and ΔL is, as a measure of its deformation, the resulting change in the length of the spring. See Figure 16.1.

While one can always choose to make this definition for a single spring, it is conceivable that it might fail to be consistent if you apply the same forces to different springs. If force A compresses a bed spring twice as much as force B, does that guarantee that force A also stretches a rubber band twice as much as force B? Is force A twice B for anything springy? Centuries of experience give us the answer, "Yes, within limits."

Hooke's model has been very successful, but two drawbacks prevent it from being considered universal. The first is that if real springy things are stretched or compressed too far, the consistency of Hooke's model between different springs starts to fail. Indeed, real springs will eventually break. The second drawback is that there exist a few oddball springy things that *never* fit the model. But avoiding those oddball "springs" and keeping ΔL small-to-moderate is not a severe restriction.

The same force will deform different things by different amounts. For a given spring, the ratio of force to resulting displacement describes how difficult it is to deform the spring. Rewriting the ratio gives **Hooke's Law**, traditionally written as

$$\frac{F_{\text{by}}}{\Delta x} = -k \quad \Rightarrow \quad F_{\text{by}} = -k\,\Delta x \quad . \tag{16.2}$$

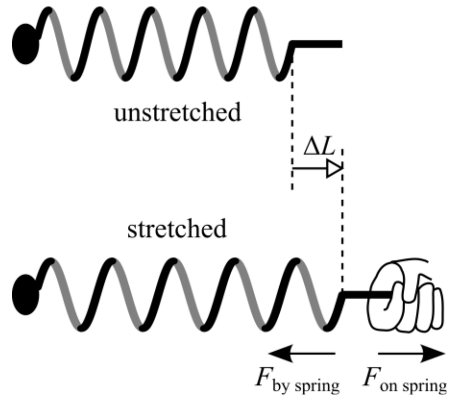

Figure 16.1
Spring before and while it is stretched.

Here k is called the **spring stiffness constant,** often shortened to **spring constant.** The new quantities need new units to be defined. The SI unit for force is the newton (N), which is roughly equal to a quarter pound. SI defines the newton in terms of more fundamental quantities, but that isn't needed here. (For checking calcualtions, $\text{N} = \text{kg} \cdot \text{m/s}^2$. For the reasons behind this, see Chapter 15.) Spring stiffness is not a root quantity in SI either. From the equation we can see that the derived basic unit for the spring constant is N/m.

In Eq. 16.2 the deformation ΔL has been replaced by the displacement Δx of one end of the spring. With the assumption that the other end is held unmoving, the two are exactly equal, so this is a change in perspective, not in the math. It is handy, because most often we are interested in the displacement of some object that is attached to the spring. But keep in mind that at its foundation, Hooke's Law is about deformation, not displacement. Notice that Δx refers to displacement of the spring's end from a very specific place, namely from the position where no external force is acting to deform the spring. This is called the **equilibrium position** for the end of the spring.

For a stretched spring, there are two forces available to focus on. One is the force applied to the spring, for instance by the hand in Figure 16.1, which was the F_{on} in Eq. 16.1. The other is the force with which the spring is pulling back, for instance on the hand, which is the F_{by} in Eq. 16.2. Hooke's Law is usually written with the second option in mind, in order to keep the definition all about the spring, regardless of what is stretching it. It turns out that F_{on} and F_{by} are always equal in magnitude and exactly opposite in direction (that is Newton's Third Law, which won't get much attention in this book), so that the proportion in Eq. 16.1 holds for either force.

Because a spring always wants to return to its undeformed state, the position-dependent force that it provides is called a **restoring force.** F_{by} and Δx will always be in opposite directions. In our treatment of physics along one axis this means they have opposite signs, making the ratio in Eq. 16.2 negative. The minus sign on the right side makes that negative explicit, so that the spring constant k will always be positive.

Figure 16.2 shows how this restoring characteristic means that a force-displacement graph only passes through the top-left and bottom-right quadrants. For springs that obey Hooke's Law, this graph forms a straight line, so that the force is called a **linear restoring force**.

Unstretched coiled springs often have their coils touching each other, so that the spring can be stretched but not compressed. But some coiled springs, along with many other springy things, can be deformed in either of two directions from equilibrium, such as either stretched or compressed. In those situations, it's not obvious that the difficulty of deformation (that is, the spring stiffness k) will be the same in both directions. Nevertheless, that is the case for the large majority of springs. Graphically, this means that a graph like Figure 16.2 has the same slope on both sides of the origin.

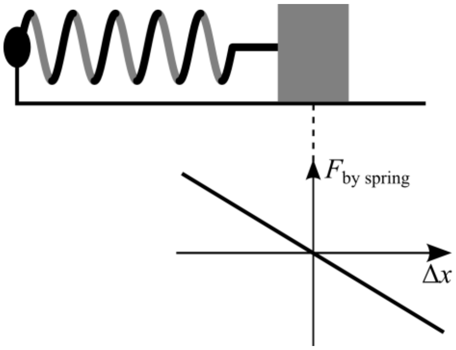

Figure 16.2

Object (gray rectangle) attached to spring, and graph of the linear restoring force that the spring applies to the object.

Chapter 17. Force Laws

Which model is better, Newton's or Hooke's? It turns out that we don't have to choose, because Newton's and Hooke's definitions of force are usually consistent with each other. The only situations in which they are not consistent are cases in which Hooke's Law fails to be internally consistent anyhow, and therefore Newton's Law is considered more fundamental. But they are both extremely important to physics, in a wide variety of situations.

It is Hooke's model that provides the most practical way to measure forces. This is because it is much easier and more convenient to measure the displacement of the end of a spring than to measure the acceleration of a massive object. The majority of devices for measuring forces (including most common weight scales, which essentially measure a force due to gravity) operate by deforming a springy object.

You will notice that both models have been elevated to the status of "law." There is no hard and fast rule about what it takes for a model to be considered a scientific law. It does *not* mean that the model is always true — Hooke's Law has its limitations. It does mean that the model has proven to be very, very useful, which would only happen if the model were often very accurate. A law is also usually pretty fundamental, that is, simple and based more on observations than on logical reasoning.

Chapter 18. Practical Forces

18a. Multiple Forces

Sometimes an object is pushed or pulled by several forces. Observations have shown that the resulting motion of the object, in particular its acceleration, is the same as if it were experiencing a single force that equals the sum of the individual forces. That imagined total force is called the **net force,**

$$F_{\text{net}} = F_1 + F_2 + F_3 + \cdots \quad . \tag{18.1}$$

This model is considered a part of Newton's Second Law, so that in Eq. 15.2 the F should really be F_{net}.

The sum here must include the directionality of the vector forces. When several children pull in different directions on a toy, it doesn't accelerate very much in any direction (unless it breaks!). For our description of one-dimensional motion, this means that some of the individual forces may be negative. The best way to handle this is *not* to change some of the additions in Eq. 18.1 into subtractions, but rather to include the negative within the variables. The distinction is subtle, so here is an example. Suppose that a rock is hanging motionless on a string, and we focus on the forces applied to the rock. The upward force of the string is balanced by the downward pull of gravity. With the usual choice of upwards as positive, the math might be

$$F_{\text{net}} = F_{\text{string}} + F_{\text{grav}} = (3\text{ N}) + (-3\text{ N}) = 0 \quad . \tag{18.2}$$

Notice that the negative is inside the parentheses.

Whenever it works to add things together to get the total effect, physicists call it **superposition.** Perhaps this gets such a grandiose name because the addition is often of things fancier than simple numbers. For instance, here we are dealing with superposition of vectors. In any case, superposition is a useful model that will show up multiple times, for instance in Chapter 37.

18b. Gravitational Forces

A hanging rock is actually very useful.

In Chapter 14, it is noted that near the earth's surface all large, dense objects in free-fall are observed to accelerate downwards at a certain rate g. Newton's Second Law then implies that any such object is feeling a force which causes that acceleration, a force due to the phenomenon of gravity. Since the acceleration is always the same, Newton's Second Law becomes a relationship between mass and this force,

$$F_{\text{grav}} = -mg \quad . \tag{18.3}$$

The magnitude of this force of gravity $\left|F_{\text{grav}}\right|$ is called the object's **weight** (although there are a few minor disagreements between physicists over precisely what the word "weight" means). Weight and mass are so closely related by Eq. 18.3 that in everyday life it is rarely necessary to distinguish between them. However, they are not exactly the same concept. If an object were to move far from the earth's surface, it would still have an unchanged mass. But if it were dropped back towards earth, it would no longer accelerate at the rate g, and Eq. 18.3 would not be correct.

The last paragraph only really considered objects in free-fall. However, there has never been any indication that the force on an object due to gravity depends on how the object is moving. Even stationary objects experience the same gravitational force given by Eq. 18.3, and this provides an extremely practical way to produce well-known forces. Just assemble some rocks of well-known mass and hang one from some support, like a string.

1. If the rock is stationary, then it is not accelerating: $a = 0$.
2. Eq. 15.2 therefore tells us that the net force on the rock is zero: $F_{\text{net}} = ma = 0$.
3. Eq. 18.1 therefore tells us that the forces from the string and from gravity have equal magnitude: $0 = F_{\text{net}} = F_{\text{string}} + F_{\text{grav}}$.
4. Eq. 18.3 tells us the magnitude of the force from the string: $F_{\text{string}} = -F_{\text{grav}} = mg$.
5. Finally, Newton's Third Law tells us that if the string is applying $F = mg$ upwards on the rock, then the rock is pulling downward on the string (or other support) with a force of the same magnitude.

Thus, you can create well-known forces by hanging well-known masses. Of course, this force is always downwards, but by running the string over a pulley, we can create a string-pulling force in any direction we like.

Chapter 19. Hanging Spring

There are many kinds of springs. But the type that most people think of first is a coiled spring, and most coiled springs are tightly coiled, so that they can be stretched but not compressed. Nevertheless, they can allow for motion to either side of an equilibrium point if an object is hung from the spring, as illustrated on the right side in Figure 19.1. With nothing on the spring, its lower end is at position y_0. When stretched, the displacement of the end is

$$\Delta y = y - y_0 \quad , \tag{19.1}$$

which is negative.

When an object is hung from the spring, gravity pulls down on the object with a constant force, while the spring pulls up on the object with a force that depends on the object's position. So, if the object is lowered gently, it will hang motionless with the spring stretched just enough to balance the gravitational force. The new equilibrium position y_1 is determined by the fact that the forces on the object cancel each other

$$\begin{aligned} -mg = F_{\text{grav}} &= -F_{\text{spring}} = -(-k\,\Delta y) \\ &= k(y_1 - y_0) \quad . \end{aligned} \tag{19.2}$$

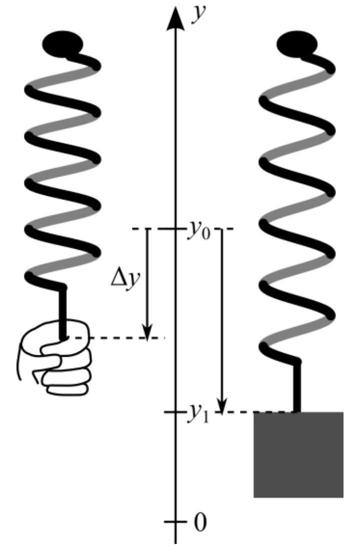

Figure 19.1
Spring being stretched down from equilibrium position y_0.

If the object is then pulled below this new equilibrium position, then the spring force increases to be larger than the gravitational force, pulling the object up. If the object is moved above its equilibrium position then the spring force decreases, and the larger gravitational force pulls the object down. In fact, the total force on the object is now

$$\begin{aligned} F_{\text{grav}} + F_{\text{spring}} &= -mg - k(y - y_0) \\ F_{\text{net}} &= k(y_1 - y_0) - k(y - y_0) \\ F_{\text{net}} &= k(y - y_1) \quad . \end{aligned} \tag{19.3}$$

where a substitution has been made in the second line using Eq. 19.2. Thus, we have a new result: this mass-and-spring system behaves as if gravity is not involved at all! Now that y_1 is the new equilibrium position for this spring, the spring can be either compressed or stretched away from that equilibrium.

While being able to solve questions with numerical answers (like the one in Figure 9.1) is nice, this conclusion about hanging springs is the kind of thing that a physicist really likes. By carefully combining a few equations, we find a result that applies in more than just one specific situation, and the result simplifies how we can think about those situations. Best of all, it is a little surprising, in that the spring stiffness constant is completely unaffected by the change in equilibrium.

Chapter 20. The "Circle of Physics"

20a. Velocity Graphs

Of course, anything that changes with time can be described with a time graph. In particular, we could graph (instantaneous) velocity versus time. This is more abstract than a position-time graph. Given a velocity-time graph, it is quite a bit more difficult to understand what motion it represents.

For instance, it is important to recognize that the motion described by Figure 20.1 does not reverse direction at any point. For the early and late times the velocity is $v = 0\,\text{m/s}$, which means that the object is stationary. In between, the object moves in a positive direction at constant speed. Nowhere is the velocity negative.

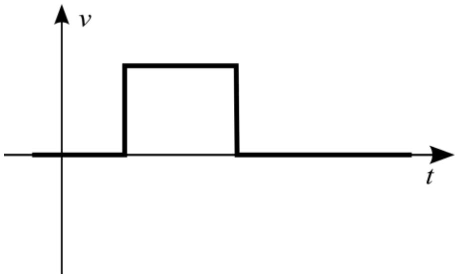

Figure 20.1
Graph of velocity versus time, with velocity always positive.

Notice that a velocity graph is missing one piece of information that is not missing for position graphs: it does not tell anything about the starting position. However, given a velocity graph and the physical position at the time origin (or, really, any particular time), it is possible to reconstruct the complete position versus time, at least in principle. On the other hand, while the origin of a position axis needs to be associated with a physical location, the origin of a velocity axis needs no clarification; it just means "not moving."

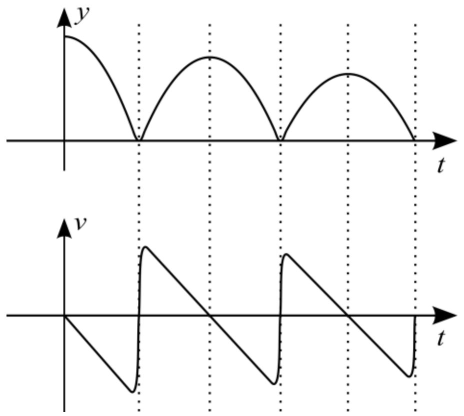

Figure 20.2

Position (height) and velocity graphs of ball bouncing on the ground.

Going the other way is a bit easier. Given a position-time graph, we can obtain the velocity at any point by finding the slope. Thus, it is possible to find the entire velocity-time graph for a motion from its position-time graph.

For instance, Figure 20.2 shows two graphs describing the motion of a ball bouncing on the ground. First, consider the position-time graph. Notice that although the graph looks similar to what you would see if a ball bounced across a room, that is not necessarily the motion it describes. The graph only tells us the vertical position of the ball (described by the variable *y*), with the floor chosen as the *y* origin. There is no horizontal position information. The ball might be bouncing straight up and down. In fact, that would be nice, since it's most similar to oscillating motions that we are interested in.

The velocity-time graph has been positioned so that its time axis lines up with the time axis from the position-time graph. The velocity is positive while the ball is moving up, and negative while the ball is moving down. At the bounces, a corner in the position-time graph indicates that the velocity changes suddenly. As a result, the velocity-time graph has an almost-vertical line at that point.

> The general rule for creating a velocity-time graph from a position-time graph is **slope to value**.

The following are some useful steps to follow when implementing that general rule.

1. Mark times where the position-time graph has zero slope (that is, it is horizontal).
2. Between the zero marks, label the time intervals according to whether the position-time slope is positive or negative. This tells you where the velocity-time graph should be above or below the horizontal axis.
3. Look for where the position-time graph is steepest. At those points in time, the velocity-time graph should be farthest from the horizontal axis.
4. Look for places where the position-time graph is a straight line. Over those time intervals, the velocity is constant, so the velocity-time graph should be horizontal.
5. If the overall motion keeps returning to the same position, then the object must experience an equal "amount" of positive and negative velocity. Although the details are beyond the scope of this book, the consequence is that if the space between the velocity curve and the horizontal axis were colored in, then the colored areas above and below the axis must be equal.

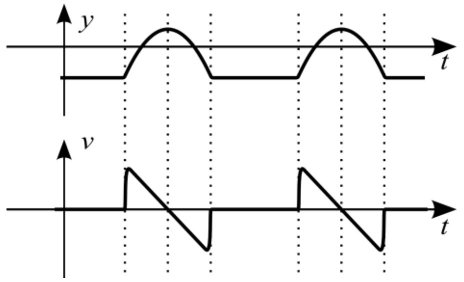

Figure 20.3

Graphs of frog jumping, with pauses between jumps.

Often the places of zero velocity found in step 1 are points in time. Sometimes, however, there are whole time intervals with no motion. Figure 20.3 shows an example, which might represent the height of a frog as it jumps. Notice that in this graph, the *y* position (or height) origin was not placed at the ground. The ground is a natural choice, but it is not required. Also notice that during the times with no motion, the rule is still "horizontal position graph gives zero on velocity graph."

20b. Acceleration Graphs

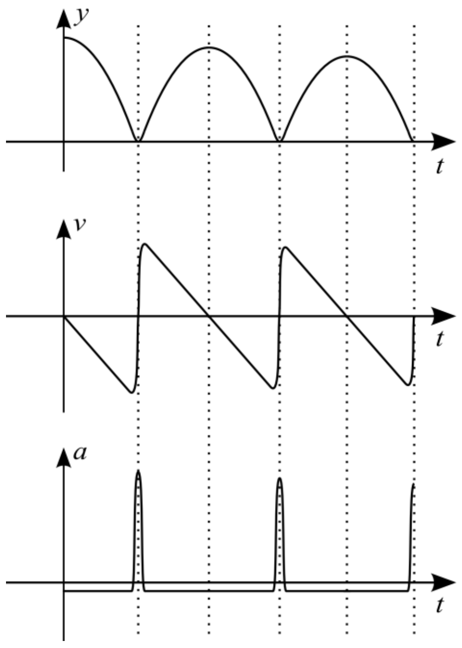

Figure 20.4

Position, velocity, and acceleration graphs of a bouncing ball.

As with position and velocity, you can graph acceleration of an object versus time as well. And since the defining relationship between acceleration and velocity (Eq. 14.1) has exactly the same pattern as the defining relationship between velocity and position (Eq. 12.1), everything in Chapter 13 and Section 20a has an analogous rule for acceleration. Once again, slope of a line on a velocity-time graph gives acceleration. If the line connects two points on the velocity-time graph, then its slope is the average acceleration between those points. But it is more useful to define **instantaneous acceleration** as the slope of a tangent line at a single point on a velocity-time graph. Once again, an acceleration-time graph can be obtained by applying the slope-to-value rule to a velocity-time graph. Figure 20.4 extends the example of a bouncing ball to an acceleration-time graph.

In the abstract, given a position-time graph, you could now find the corresponding acceleration-time graph. First make the velocity graph, and then all the steps from Section 20a apply for getting the acceleration graph from the velocity graph. But in practice, trying to do slope-to-value twice in a row can get sloppy. To get cleaner and more accurate graphs, it helps to consider the following additional clues.

1. If the moving object is ever in free-fall, then the acceleration is known to be downwards and constant, in particular $a = -g = -9.8\,\mathrm{m/s^2}$. Figure 20.4 has examples of this. Working backwards, this means that the velocity-time graph is a straight line with negative slope.

   If the moving object is *not* in free-fall, then cases of constant acceleration are rather rare.

2. Sharp corners in the position-time graph indicate brief intervals of time with large "pulses" of acceleration.

3. Considering the forces that the object feels may be helpful. Because of Newton's Second Law, the direction and relative magnitudes of forces will be the same as those of accelerations. That is, a graph of the net force on the object versus time would look exactly the same as the acceleration graph, although the scale on the vertical axis would be different.

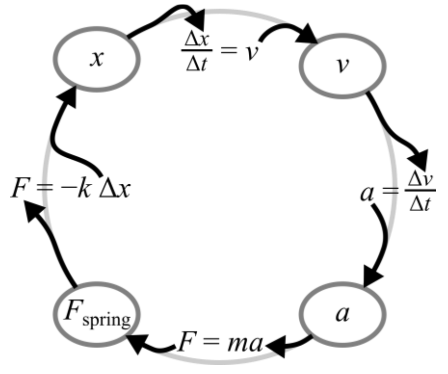

Figure 20.5

Circle of physics equations.

20c. The "Circle of Physics"

Attaching an object to the end of a spring that can be both compressed and stretched, as in Figure 16.2, applies a restoring force on the object, with the result that the object can oscillate. This is why Hooke's Law is particularly important to the subject of sound. It is true that a restoring force does not have to be linear in order to cause oscillation. But it is a very common situation. In this situation, exactly what sort of motion will occur?

Combining our equations from kinematics and dynamics, we find that the four main variables are linked by four equations, as illustrated in Figure 20.5. The figure uses the assumption that the displacement of the object and the stretch or compression of the spring are exactly the same, $\Delta L = \Delta x$. The variables Δx, v, and a all refer to the object, while F refers to the force applied on the object by the spring.

When equations link variables in a "circle of physics" like this, physicists get very excited. It means that only certain types of motion can satisfy the equations. When solving a numerical question, as in Chapter 9, finding four equations relating four unknowns would mean that you have found enough information to answer the question. The current situation is similar, except that some of these equations involve changes over time. As a result, the "solution" is not just a number; the solution is an entire type of motion, that is, a particular way that position varies in time.

Calculus is necessary to derive exactly the kind of motion that works, but graphing techniques can well illustrate how the equations constrain the possible motions. Our goal is to find a displacement-time graph such that, after obtaining velocity-, acceleration-, and force-time graphs, we will come back to the same position-time graph after working all the way around the circle of physics.

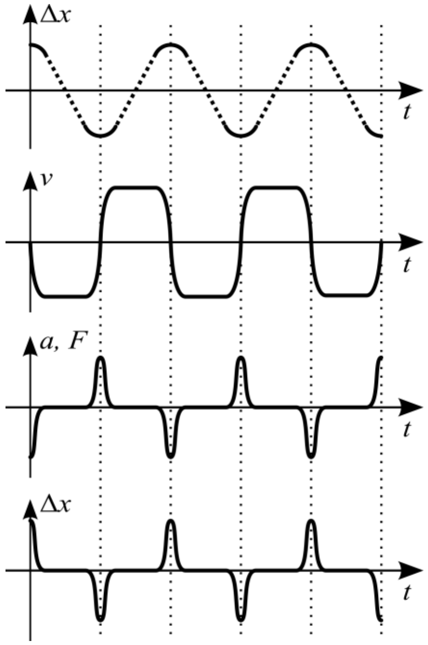

Figure 20.6

An oscillation that fails to satisfy circle of physics, because bottom Δx graph doesn't match top graph.

For instance, the top of Figure 20.6 shows a plausible motion for this oscillation. It is constant velocity motion (the dotted lines) alternating with reversals of direction (the solid curves). In the next two graphs, slope-to-value is used to obtain the corresponding velocity and acceleration graphs. A graph of the force F would look the same as the graph of a, because Newton's Second Law tells us that they are proportional with the same sign. Finally, Hooke's Law tells us that displacement should be proportional to the force, but with the opposite sign.

Since the bottom graph of Figure 20.6 does not match the top graph, this motion *does not* satisfy the circle of physics. The most significant failing is that the straight segments in the top graph result in intervals of zero in the bottom graph.

The displacement-time graph that does successfully satisfy the circle of physics is illustrated in Figure 20.7. Graphs of this shape are called **sinusoidal graphs** because the mathematical functions that generate them are the sine or cosine functions.

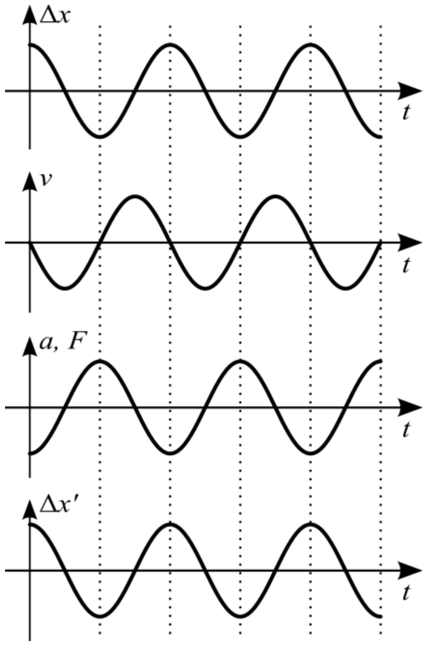

Figure 20.7

Oscillation that does satisfy the circle of physics.

This particular, extremely common natural motion, arising from a linear restoring force, is named **Simple Harmonic**

Motion, or SHM for short. SHM and sinusoidal motion more generally are extremely important to the study of sound.

Chapter 21. Sinusoidal Motion Parameters

Sinusoidal motion, also called simple harmonic motion or SHM, is of fundamental importance to the study of sound. The shape of this motion is described by either the sine or cosine function; in this book, cosine will be used. The algebraic equation relating displacement to time is

$$\Delta x = \blacksquare \sin(\blacksquare t + \blacksquare) \quad \text{or} \quad \Delta x = \blacksquare \cos(\blacksquare t + \blacksquare) \quad . \tag{21.1}$$

In these equations, black squares indicate where additional numerical quantities must be added to fill in details and to make the units work out. A particular sinusoidal motion is described by just a few of these **parameters,** as illustrated in the position-time graph of Figure 21.1.

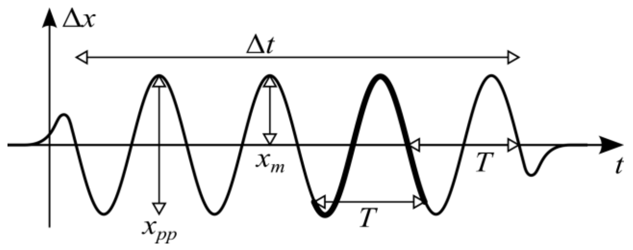

Figure 21.1
Graph illustrating the parameters of SHM.

Time Parameters

SHM is a repeating motion, and the parameters from Chapter 10 apply to describe the "size of the motion" along the time axis. The sine and cosine mathematical functions describe curves that have no beginning and no end, so their **duration** is infinite. Figure 21.1, like real examples of SHM, has a finite duration indicated by Δt, although precisely when the repeating motion begins and ends is not perfectly well defined.

The most obvious way to measure the **period** is from one maximum (or minimum) to the next. But on a drawn graph, intersections can be more precisely identified. Since a **cycle** can start anywhere (an example is highlighted as a heavy curve in the figure), any horizontal line can be used to measure the period between intersections. Just remember to skip one intersection, where the motion is in the opposite direction (upward or downward). The most convenient horizontal line is likely to be the time axis.

Amplitude

Simple harmonic motion oscillates to either side of the **equilibrium position** $\Delta x = 0$, a place where the moving object could indefinitely rest stationary without any external influence. The maximum degree to which the moving object is displaced from equilibrium is the **amplitude**. In the field or lab, it's often the most practical to measure the distance from one extreme of the motion to the other. This is called the **peak-to-peak amplitude**, labeled in the figure as x_{pp}. When working mathematically, however, it is usually more convenient to refer to the distance from the equilibrium to either of the extremes, which is simply called the **amplitude** (no preceding adjective) and is labeled in the figure as x_m. The subscript *m* is short for "maximum." The variable A is also commonly used for amplitude, but it will not be used in this book due to conflicts with other concepts.

Notice that amplitude and displacement are not the same thing. This pair of concepts are so closely related that many students find them difficult to keep straight. Since amplitude equals the *maximum* displacement, they both have the same root unit of meters.

Phase

Finally, there is a parameter of sinusoidal motion that is in some sense less fundamental. In Figure 21.2, graphs a) and b) show *motions* that are identical (assuming that the two time axes are synchronized). But the *graphs* differ in one respect: they have different choices for the moment in time designated as $t = 0$. On the other hand, graphs b) and c) show motions that really are different, in that one is shifted along the time axis.

To describe these things, we start with the idea of **phase**. Phase is a way to describe how far through a cycle some point is. Since a cycle can start anywhere, we first need to choose a convention for where a cycle starts. Throughout this book, we will choose the convention of the cosine function, which is that a cycle starts at a maximum, which therefore has a phase of zero. Each cycle is then divided into 360 degrees (symbol °) that are equally spaced along the time axis, with numbers increasing in the same direction as increasing time. The resulting numbers measure phase along the sinusoidal graph, usually represented algebraically by the symbol ϕ (the Greek letter phi).

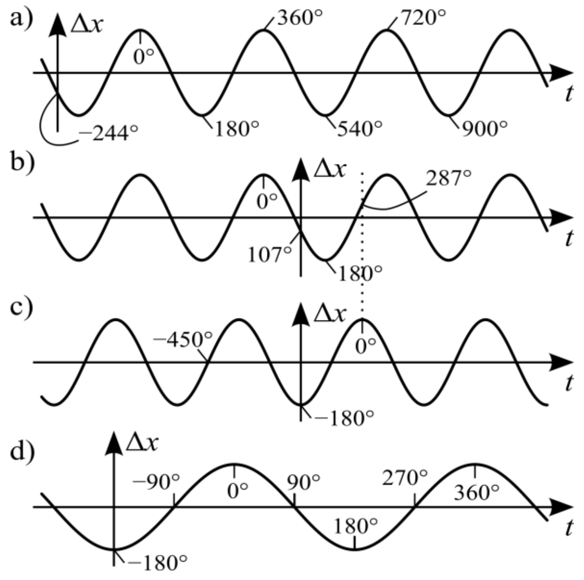

Figure 21.2

Graphs illustrating phase in SHM.

Several phases are indicated in Figure 21.2. Notice that, although the degrees indicate equal spacing along the horizontal axis, the phase is associated with the curve, not the axis. Also notice that there is no rule about *which* maximum should be 0°, so that there is some built-in ambiguity as to how to label the graph. Phases $\phi = 0°$, $\phi = 360°$, and $\phi = 720°$ all mean the same thing, and $\phi = -90°$ is the same as $\phi = 270°$. You can always add or subtract 360° without changing the meaning. It is therefore sometimes convenient to convert all phases into the range 0° to 360°; in this book, that will be called the **reduced phase** when we need to be explicit.

Having the degrees of phase equally spaced along the time axis is another way to say that changes of phase are proportional to changes of time, $\Delta\phi \propto \Delta t$. On a given sinusoidal motion, a pair of points defines a time difference and a phase difference; the ratio of those differences will be the same, regardless of which pair of points you choose. One particular choice to consider is points separated by one cycle, making the proportion equation

$$\frac{\Delta\phi}{\Delta t} = \frac{360°}{T} \quad . \tag{21.2}$$

The proportion means that the left side of that equation can refer to *any* pair of points; only the right side is specifically for one cycle.

Using the phase idea, we can specify how a sinusoidal graph relates to the time origin by specifying the phase which occurs at $t = 0$. This is called the **initial phase** or the **phase constant** and is usually denoted by ϕ_0. The initial phase of sinusoidal motion is less fundamental than the other parameters because it depends on when we choose the time origin, rather than on the physical reality. In Figure 21.2, the top two graphs show the same physical reality but have different initial phases.

However, the phase itself does have physical meaning. The physically real shift between graphs (b) and (c) of Figure 21.2 can be described by their **phase difference:** at any moment in time, subtract the phases to get $\Delta\phi = 287°$. This works no matter which moment in time you choose. Because the time origins of the two graphs match, the easiest choice is to use the initial phases, so that the phase difference can be calculated as the difference of the initial phases, $\Delta\phi = \phi_{0b} - \phi_{0c} = 107° - (-180°) = 287°$.

As a final note, it is no accident that phase and angle are both measured in degrees. Sinusoidal motion repeats every cycle just as the angle between a clock's minute hand and vertical repeats ever hour. In fact, a graph of the vertical position of the minute hand's point would be precisely sinusoidal motion. On the other hand, the fact that temperature units are also called degrees *is* an accident of etymology.

Chapter 22. Sinusoidal Motion Function

The algebraic function that describes the shape of SHM is cosine (or sine). That is, this function allows the calculation of vertical information (e.g., displacement) from horizontal information (time). The correct formula for a specific simple harmonic motion contains all the information about it, so in theory the formula can be used to answer any question about SHM. In practice, however, this often is not the easiest way to answer a question, so do not consider the following to be a panacea.

Given the phase ϕ at some point in the oscillation, one can calculate the displacement using the formula

$$\mathit{\Delta} x = x_m \cos\phi \quad . \tag{22.1}$$

The phase, in turn, can be related to time through Eq. 21.2. By choosing the time origin as the reference point for the differences in the left side of that equation, it becomes

$$\frac{\phi - \phi_0}{t - 0} = \frac{360°}{T} \quad , \tag{22.2}$$

which together with Eq. 22.1 yields the basic equation for sinusoidal motion that fills in the gaps in Eq. 21.1:

$$\mathit{\Delta} x = x_m \cos\left(\frac{360°}{T} t + \phi_0\right) \quad . \tag{22.3}$$

Chapter 23. Importance of SHM

There are a number of reasons why SHM is so important to the study of sound. Here are a few.

- Sinusoidal motion causes sounds that pleasant to hear, or at least not harsh. In fact, a sound produced by sinusoidal motion is called a **pure tone**.
- Sinusoidal functions are relatively easy to work with mathematically.
- An object attached to a spring creates a **system** that moves in a way that is well modeled by SHM. If you have read Section 20c, you have seen how SHM satisfies the circle of physics. In brief, the spring's resting shape defines an equilibrium, and deforming the spring provides a **linear restoring force.** Springy things are very common in the real world, and SHM is their **natural motion.**
- Even in cases where springy things provide a restoring force that is not linear, SHM is often a very good approximation for the resulting motion. This is especially true if the amplitude is small. (As detailed in Chapter 16, this is precisely because a straight line is a good approximation for a very short piece of a curving force-versus-displacement graph.) SHM can also be a good starting place from which to describe the actual motion with a non-linear restoring force, although that is beyond the scope of this book.
- But possibly the biggest reason that SHM is important can't be described until Chapter 42.

A sinusoidal function goes on forever, with the amplitude always remaining the same. This is certainly not true of real oscillations, which all decay in size if observed for long enough. Given that, why should simple harmonic motion remain so important? One reason is that, for oscillations whose amplitude decays slowly enough, SHM is a good model of the motion for comparatively short intervals of time. Another is that, as covered in Chapter 26, several properties of oscillation are unaffected by changes in amplitude.

Chapter 24. SHM Speed

As an object executes SHM, how fast does it go? As the object moves from a minimum (that is, most negative) to a maximum of displacement, since there is no change in direction of travel, the average speed is the magnitude of the average velocity. The average velocity is equal to the slope of a line connecting those points on a displacement-time graph,

$$s_{avg,pp} = \left|v_{avg,pp}\right| = \frac{\Delta x}{\Delta t} = \frac{2x_m}{\frac{1}{2}T} = 4\frac{x_m}{T} \quad . \qquad (24.1)$$

However, that is not the maximum speed achieved during the motion. The steepest point on the sinusoidal graph is where the displacement is zero. It turns out that a tangent line at that point has the slope

$$s_m = \frac{\pi}{2} s_{avg,pp} = 2\pi \frac{x_m}{T} \quad . \qquad (24.2)$$

Deriving that result is most easily done with calculus, and thus is beyond the scope of this book. But it should be apparent on a graph that the maximum slope is roughly 50% higher than the average found in Eq. 24.1.

Chapter 25. Sinusoids Beget Sinusoids

When the displacement of an object varies sinusoidally with time, it turns out that many other characteristics of the motion also vary sinusoidally with time as well. For instance, the velocity of the object, the acceleration of the object, and the force applied on the object are all sinusoids, as is apparent in Figure 20.7. We will see other sinusoids cropping up.

When the graph of any quantity versus time has a sinusoidal shape, it can be described by essentially the same parameters as are used in Chapter 21. The concept of period is completely unchanged. The amplitude idea is adjusted to whatever is on the vertical axis. For instance, for a sinusoidal velocity-time graph, one might refer to the maximum velocity achieved as the **velocity amplitude** v_m. The phase idea is mostly unchanged; the only possible modification is that it might be desirable to choose a different point on the sinusoidal shape as $\phi = 0°$.

The equations in Chapter 24 are really just relationships about slopes of the sinusoid shape. They could equally well be applied to get the rate of change for any vertical axis quantity. For instance, applied to a velocity-time graph, they can be used to find the accelerations.

Chapter 26. SHM Frequency and Amplitude

When an object has SHM as its natural motion, it oscillates at a very specific period and frequency. In particular, the frequency of the oscillation is the same regardless of the motion's amplitude. This may seem very counterintuitive. A larger amplitude motion requires traveling larger distances with each cycle, and one might well expect those to take a longer time, resulting in a longer period for the motion.

However, larger displacements also mean that the spring is more deformed, so that the forces involved are larger. These cause higher accelerations at the extremes of motion, so that the maximum speed is also larger. In the end, it turns out that these two effects (larger distances and larger speeds) exactly cancel each other out, so that the period and frequency are the same for all amplitudes.

This is quite a special property. Non-sinusoidal oscillations, called **anharmonic,** will vary in frequency if the amplitude changes.

Chapter 27. SHM Frequency Formula

When an object moves with natural SHM because it is attached to a spring, the strength of the spring and the mass of the object control the period and frequency. A stiffer spring (higher value of k) will result in a larger force (all other things being equal), which moves the object faster, resulting in a shorter period and a higher frequency. Conversely, a more massive object is more difficult to accelerate, so that (all other things being equal) the object moves more slowly, resulting in a longer period and a lower frequency. The exact relationship is

$$T_0 = 2\pi\sqrt{\frac{m}{k}} \quad , \tag{27.1}$$

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad . \tag{27.2}$$

Although these equations are offered without derivation, you should still be able to verify that they do, indeed, exhibit the relationships described above. The results are called the **natural period** and **natural frequency** of that particular system. The subscript zero is commonly used to indicate this natural status, i.e., that the variable represented arises is a property of the vibrating system with no external influence.

Chapter 28. Helmholtz Resonators

One unusual form of a mass on a spring is composed almost entirely of air. It's called a Helmholtz resonator, named after the German physicist who pioneered the use of such devices in sound analysis.[6] All that is needed is an enclosed volume that is connected to the outside by a hole or short tube. Whenever you make a hooting sound by blowing across the mouth of a bottle, or whenever you whistle, you are using a Helmholtz resonator, where the bottle or your oral cavity, respectively, make the enclosed volume. The surprising part is that these can be understood as a mass on a spring.

This book assumes that you have a good familiarity with the idea of volume. Its traditional variable is capital *V*. The SI unit for volume is meters cubed, also called cubic meters, symbolized by m^3.

The details of how a Helmholtz resonator works will be covered in Chapter 135. But it is easiest to conceptualize for a bottle with a single, straight-sided neck, as shown in Figure 28.1. The air in the neck of the bottle (the gray chunk) has a mass, albeit very small. It can oscillate up and down, which causes the air in the body of the bottle to expand or compress. The air in the body acts like a spring, as described in Chapter 134, pushing the neck air back towards an equilibrium position.

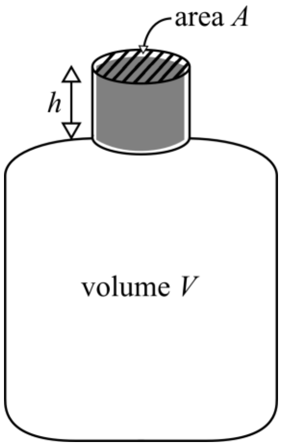

Figure 28.1
Dimensions of a bottle, considered as a Helmholtz resonator.

The end result is a system that vibrates with a natural frequency given by the equation

$$f_0 = \frac{s}{2\pi}\sqrt{\frac{A}{Vh}} \quad , \tag{28.1}$$

where s is the speed of sound for the air in the bottle, and h, A, and V are dimensions of the bottle, respectively the length of the bottle neck (along the axis of vibration), the cross-sectional area of the bottle neck (perpendicular to the axis of vibration), and the volume of the bottle body. You can see in the form of this equation hints that it came from Eq. 27.2, although exactly how the two relate has become obscured.

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[6] Hermann von Helmholtz, *On the sensations of tone as a physiological basis for the theory of music*, trans. Alexander J. Ellis (London: Longmans, Green, and Co., 1885), 43.

Beyond where Eq. 28.1 comes from, there are other complications that can arise. When whistling, your oral cavity is open at both your lips and the back of your throat—how does that change things? And why should a steady blowing over the mouth of a bottle cause it to vibrate? (See Chapter 196 for a partial answer.) In any case, the fact that it does vibrate in response to an external cause is the reason that the setup is called a "resonator" instead of a "vibrator."

You have probably experienced the Helmholtz resonator in another way, when someone opens a single window in a travelling car and a throbbing sound results. Here the cabin of the car forms the volume, and the open window forms the neck. However, Eq. 28.1 can't be applied properly because there is no apparent length to the neck. The opening in the window provides a clear area *A*, but what should you use for the variable h?

When faced with such a situation in a physics question, one usually needs to look back at where the equation came from. Certainly, it is *not* a recipe for success to cast about for another value to use. To think, "I'm missing a length… maybe it will work if I just use the neck diameter," is guessing, not science.

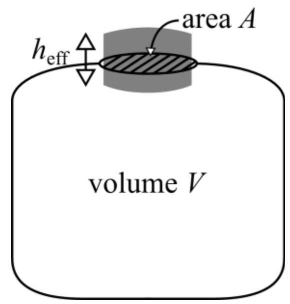

Figure 28.2
Dimensions of Helmholtz resonator with no neck.

The length of the neck was used in Figure 28.1 because that is also the length of the oscillating chunk of air. With an opening instead of a neck, there can still be an oscillating chunk of air, as in Figure 28.2, but how thick is it? Answering that question requires physics well beyond the scope of this book. So instead, use this rule of thumb: the effective thickness / neck length is 85% of the smallest width of the opening,

$$h_{\text{eff}} = 0.85 \cdot w_{\text{min}} \quad . \tag{28.2}$$

For a circular opening, w_{min} is the diameter. For an oblong opening, w_{min} is the width, not the length.

Many similar shapes act as Helmholtz resonators, even though the shape makes it difficult to say precisely how large the "neck," or even the "body," is. In fact, a detailed investigation of even a perfectly cylindrical neck will reveal that it has an effective length that is slightly longer than its physical length. Nevertheless, judicious use of the two models above can give quite acceptable results.

Chapter 29. Energy

In preceding chapters, the motion of objects has been considered in light of the forces they feel. But there is an alternative viewpoint which is sometimes easier to use, and which makes some relationships easier to understand: the viewpoint of **energy**. As a starting point, here is a qualitative definition of energy, as used in physics:

> Energy is a property possessed by a system, quantifying the system's ability to affect some other physical system.

Energy can come in different forms. The most visible form of energy is energy of motion, called **kinetic energy** (from the Greek word for motion). For example, a moving golf club can hit a ball, causing the ball to fly off. A moving hammer can shove a nail further into wood. In both cases, the kinetic energy of the moving thing allows it to affect something else. A less obvious form of energy is energy due to shape, called **potential energy**. For instance, a stretched slingshot can be released, flinging (affecting) the stone it is holding. Potential energy is often thought of as stored energy. Energy associated with visibly large objects moving or being deformed is called **mechanical energy**. Thus, some mechanical energy is kinetic, and some mechanical energy is potential.

This qualitative idea is fine as far as it goes, but it took many talented scientists several centuries during the Renaissance to figure out and implement a very special property of energy. If energy is quantified in the right way, then the total amount of energy around us is constant. Energy is not "stuff" in the same sense as objects in the material world. But it nevertheless shares some of the properties of "stuff": it can be moved around, but it cannot be created or destroyed, although it can be converted from one form to another. In technical language, energy is **conserved**.

In order to make this energy conservation idea work, from time to time physicists have had to invent (or discover, depending on your perspective) new forms of energy. For instance, a book held high in the air can be dropped, with the result that its kinetic energy increases. If energy is to be conserved, then the book must have had that energy all along, simply by virtue of being high in the air; this is another form of **potential energy**. (An even more enlightened understanding of this energy is that is belongs to the combination of the book and the earth, which are pulling each other together with gravity.) A hot pan from an oven can burn you, which is certainly affecting you, so it must have energy by virtue of being hot; this is named **thermal energy**. (It was not until the 1840s that James Joule convincingly showed that thermal energy was the same sort of thing as mechanical energy.) Several other forms of energy are known, but they are beyond the scope of this book.

This model of conserved energy has been so successful over history, that it is now considered one of the most fundamental tenants of physics, even more basic than Newton's Second Law (see Chapter 15). Every time a new form of energy has been found, the guiding principle of energy conservation has helped us to understand that new form. Sometimes it is useful to approach a physics question in a way that doesn't make energy conservation evident. For example, when a soccer ball rolls to a stop on level grass, its mechanical energy clearly is not conserved, but useful information can be obtained without identifying where that energy went. Nevertheless, you can also be assured that the energy did go somewhere, either to a different object, or into a non-mechanical form, or both.

When energy is transferred from one object or system to another in a macroscopic way, the amount of energy transferred is called the **work** done in the process. Be warned that this terminology only sometimes aligns with the English use of the word. Simply holding a heavy object stationary in the air could well be called work in the everyday sense, but in the physics sense no work is being done, because the object is not receiving any additional energy.

The SI root unit for energy is the joule, abbreviated J. A joule is not a huge amount of energy. If you drop a 5 pound bag of sugar or flour a distance of 5 cm, it gains roughly 1 J of kinetic energy. In a thermal form, it would take about 130 J to warm a thimbleful of water from room temperature to hot tap water temperature. How the joule derives from other units will be seen in Chapter 30.

Chapter 30. Mechanical Energy

How should the various types of energy be quantified, so that conservation of energy works? Some insight into the equations can be obtained by considering work in the colloquial sense of "effort." Consider, for instance, how much work must be done to lift a book into the air, thereby giving it potential energy. More massive books require more effort to lift. If gravity were weaker (for instance, by being on the moon), it would require less effort to lift anything. And the farther the book is lifted, the more effort is required. The formula for **gravitational potential energy** (near the earth's surface) indeed includes all these factors:

$$PE_{\text{grav}} = mgh \quad , \tag{30.1}$$

where m is the mass of the object, h is the height that the object has been lifted, and $g = 9.8\,\mathrm{m/s^2}$ is the acceleration due to gravity from Chapter 14.

How much work must be done to stretch a spring, thereby giving it potential energy? Again, the more the spring is to be stretched, the more effort is required. But here, there are two separate reasons for that added effort. Stretching farther requires more effort, just as with gravity. But also, the force against which you are pulling gets larger, due to Hooke's law from Chapter 16. As a result, the spring potential energy is proportional to the *square* of the distance stretched Δx,

$$PE_{\text{spring}} \propto \Delta x \cdot \Delta x = \Delta x^2 \quad . \tag{30.2}$$

Stiffer springs are also harder to stretch, so the spring stiffness constant k should show up. The final formula turns out to be

$$PE_{\text{spring}} = \frac{1}{2}k\,\Delta x^2 \quad . \tag{30.3}$$

Notice that the square means that spring potential energy is always positive, whether the spring is stretched or compressed. This correctly corresponds to the observation that one must do work on a spring to deform it either way. Notice also the $\frac{1}{2}$, which tends to show up in equations that have quantities squared.

The equation for kinetic energy has a similar form to that of Eq. 30.3, but it is harder to rationalize because it involves velocity v instead of position. The equation for an object with mass m is

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}ms^2 \quad . \tag{30.4}$$

Because the velocity is squared, the kinetic energy is again positive regardless of the sign (that is, the direction) of the velocity. This allows for the second form, using speed s instead of velocity. Even when it is necessary to specify the direction of velocity with a fully-fledged vector, the square of a vector is defined so that it is a positive number.

As usual, these equations tell us not only how to quantify energy, but also how the SI units relate. Working from Eq. 30.4, we have

$$1\ \text{J} = 1\ \text{kg}\left(\frac{\text{m}}{\text{s}}\right)^2 = 1\ \text{kg}\frac{\text{m}^2}{\text{s}^2} \quad . \tag{30.5}$$

Notice that a 1 kg object moving at $1\ \text{m/s}$ does *not* have 1 J of kinetic energy, but rather 0.5 J. When doing the calculation, the numbers combine separately from the units.

This book always uses a variable for energy that contains a capital E. Other references, in an effort to avoid two-letter variables, use U for potential energy and T for kinetic energy.

Chapter 31. Energy in SHM

Ideal SHM never dies away. This together with the conservation of energy imply that the total mechanical energy in the system is always the same. But energy is continually traded back and forth between object and spring. It's also continually changing type, between KE (of the object) and PE (in the spring),

$$E_{\text{tot}} = PE_{\text{spring}} + KE_{\text{object}} = \text{ a constant} \quad . \tag{31.1}$$

For instance, at the moment in time represented by the open circle in Figure 31.1(a), the displacement-time graph is horizontal, so that the velocity and kinetic energy of the object are both zero. But the spring is stretched, so that it is storing potential energy. In fact, both the stretching and the PE are at a maximum.

As time progresses, the spring uses that potential energy to do work on the object, pulling it toward the equilibrium position. By the time ¼ period has passed (filled circle in Figure 31.1(a)) the displacement is $\Delta x = 0$, so that the spring has used up all of its energy. The object, however, has reached its maximum speed, and therefore its maximum kinetic energy.

As the mass continues to move, it starts to do work on the spring by compressing it. Eventually, all the energy is transferred into potential energy of spring compression, and the object once again has zero velocity. At this point, half a cycle past where this description started, the energy situation is almost the same as when we started. The only difference is that the spring potential energy is due to compression instead of stretching.

These observations qualitatively explain the energy graphs in Figure 31.1. Notice that the energy graphs repeat their shape twice as frequently as the displacement graph. A surprising fact is that, for sinusoidal motion, the shapes of the energy graphs are *also* sinusoidal (although they are shifted up so that they are always positive). The bottom graph in Figure 31.1 shows how these two sinusoids fit together to make the total energy constant; think of flipping the KE graph upside-down to make the gray part of the E graph.

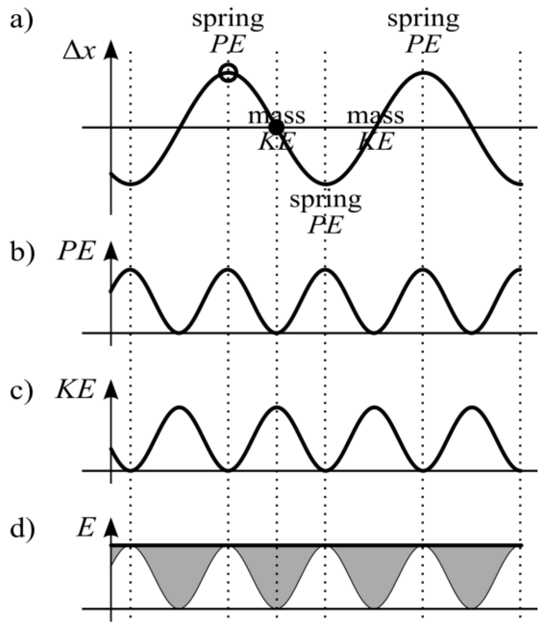

Figure 31.1
Displacement and energy in SHM. In bottom graph, constant total energy is broken into potential energy (white) and kinetic energy (gray).

Consider the energy balance at two specific times. At the open circle in Figure 31.1(a), since $KE = 0$, we can write

$$E_{\text{tot}} = PE_{\max} + 0 \quad . \tag{31.2}$$

On the other hand, at the filled circle in Figure 31.1(a), the spring has no potential energy, so that all of the energy is in kinetic form, leading to

$$E_{\text{tot}} = 0 + KE_{\max} \quad . \tag{31.3}$$

Equation 31.1 tells us that the total energy E_{tot} is the same at both times, so that we may set Eq. 31.2 and Eq. 31.3 equal to obtain

$$E_{\text{tot}} = PE_{\max} = KE_{\max} \quad , \tag{31.4}$$

which might look like it contradicts Eq. 31.1. But it doesn't because of the subscripts.

We can use Eq. 30.3 and Eq. 30.4 to express this energy equality in terms of displacement and velocity. But we must be careful, because we want to refer to the displacement and the velocity at different times. Happily, we have already defined variables for these specific quantities. The maximum PE results from the maximum stretch of the spring, which is the amplitude of the motion, and the maximum KE results from the maximum speed. Substituting those into Eqs. 30.3 and 30.4 gives the relations

$$PE_{\max} = \frac{1}{2} k\, x_m^2 \quad , \tag{31.5}$$

$$KE_{\max} = \frac{1}{2} m\, v_m^2 \quad . \tag{31.6}$$

All of this allows us to discover the following

$$E_{\text{tot}} = \tfrac{1}{2}\,k\,x_m^2 = \tfrac{1}{2}\,m\,v_m^2$$

$$k\,x_m^2 = m\left(2\pi\frac{x_m}{T}\right)^2 = (2\pi)^2 m\frac{x_m^2}{T^2} \tag{31.7}$$

$$T = 2\pi\sqrt{\frac{m}{k}} \quad , \tag{31.8}$$

where in the second line the maximum velocity from Eq. 24.2 was substituted.

So, by combining basic energy ideas, we have derived the special property of SHM that is given with no justification in Chapter 27. This is an example of the power of the energy viewpoint. To obtain the same result considering only forces and accelerations would be far more difficult.

This derivation is also an example of the power of physics in general. When reading about this formula in Chapter 27, it probably seems to come out of nowhere. At that point, there is no reason to expect squares, let alone square roots, in the equations relating to SHM. But now we have derived that formula using relatively simple ideas about energy. In physics we often find that relatively straightforward relationships can lead to surprisingly complex behavior or can imply surprisingly complex relationships.

Chapter 32. Energy and Amplitude

32a. Energy in Non-Natural Sinusoidal Motions

An object can move sinusoidally in time for reasons other than being attached to a spring. For instance, a person might be holding it, and shaking it back and forth. Or, the object might be attached to something more mechanical, such as a motor designed to push and pull the object. This is not natural motion, because an external agent is causing it. Call it **driven motion.**

If the driven motion is sinusoidal, then the object's kinetic energy behaves the same as if it were on a spring. Figure 31.1(a) and (c) would still apply. However, with no spring to hold potential energy, conservation of energy is no longer helpful in describing the motion. Instead of energy being stored in a spring and then returned to the object, energy is continually transfering to and from the driver in a way that is independent of the motion itself.

Even if an object is subject to a spring-like restoring force, an external agent can drive the system in sinusoidal motion. An object could be attached to both a spring *and* a motor. But the driving force might oscillate at a frequency that's different from the natural frequency of the spring and mass alone. Again, the kinetic energy would behave as in Figure 31.1, varying at the actual frequency rather than the natural frequency. The potential energy in the spring would also vary sinusoidally. This motion differs from natural motion in that the kinetic and potential energies in Figure 31.1(b) and (c) would not have the same maximum values. As a result, the total energy would vary in each cycle.

The energy of the vibrating system is not constant, but in the bigger picture energy is still always conserved. As the object moves, the external agent is doing work on it, transferring energy to and from it. (Removing energy from the object is considered negative work.) But since the system is not self-contained, many of the conclusions in Chapter 31 do not apply.

32b. Energy and Amplitude

Equations 31.5–31.7 are examples of a very general principle. Whenever the topic is natural motion of oscillations or waves, all quantities associated with energy are proportional to the square of any quantity measuring the amplitude of the motion:

$$\text{energy-related quantity} \propto (\text{amplitude of any sort})^2 \quad . \tag{32.1}$$

The amplitude need not be the usual "maximum displacement" amplitude. In Chapter 24 and in Eq. 31.6, s_m and v_m are velocity amplitudes. Even if you do not know the equation relatting your amplitude to an energy-related quantity, you can still use this proportionality to compare two natural oscillations or waves.

What about oscillating motion that is not natural motion? You can often identify this situation if the frequency of the motion is determined by something outside the system. Then this proportion can still be useful, but it's necessary to be more careful that the energy and amplitude are directly related. For example, the maximum kinetic energy would always be proportional to the square of the velocity amplitude, because of Eq. 31.6. However, the kinetic energy might not be proportional to the maximum displacement.

Chapter 33. Damped Harmonic Motion

Oscillations in the real world decay in amplitude, sometimes very slowly and sometimes quickly. A slow decay of an oscillating system is called **damping**. The reason for damping is that forces, other than the restoring force required for vibrations, act on the moving object. The detailed nature of those forces can be complicated, but often they continuously impede the motion; friction is the most well-known example. These additional forces were not included in the model used for Section 20c, which is why the "Circle of Physics" predicted everlasting SHM.

Because the damping forces tend to impede the oscillatory motion, you might guess that damping would slow the rate of oscillation. This is indeed true. If damping forces are added to an oscillator, they will increase its period and reduce its frequency. However, this effect is generally extremely small. For example, in the specific case of DHM (described below), damping forces that are sufficient to change the frequency by only 0.1% will also cause the amplitude to be cut in half after less than 2.5 cycles! So, in most cases, it's an excellent approximation that the damped frequency is equal to the natural frequency.

One common example is a particular damped motion called **damped harmonic motion**, or **DHM**. DHM is illustrated by the solid curve in Figure 33.1. This requires a particular type of damping force. For example, rubbing friction does *not* result in DHM, while air resistance does. Many damped motions that are technically not DHM are nevertheless well modeled by DHM.

DHM preserves some of the properties from SHM. Although it is not truly a repeating motion, it clearly has a cyclical aspect, and in fact the mathematical function describing the motion contains a sinusoid. A period can be measured for those cycles. If it is measured between points at the equilibrium position, where $\Delta x = 0$, then the period and frequency stay precisely the same throughout the motion. Thus, one can say that the frequency of oscillation is still independent of amplitude. That is, although the period in Figure 33.1 is approximately the same as it would be without damping, the period is *exactly* the same on the left and right sides of the figure. The time interval between pairs of maxima or pairs of minima is not exactly equal to the period, although it is very close.

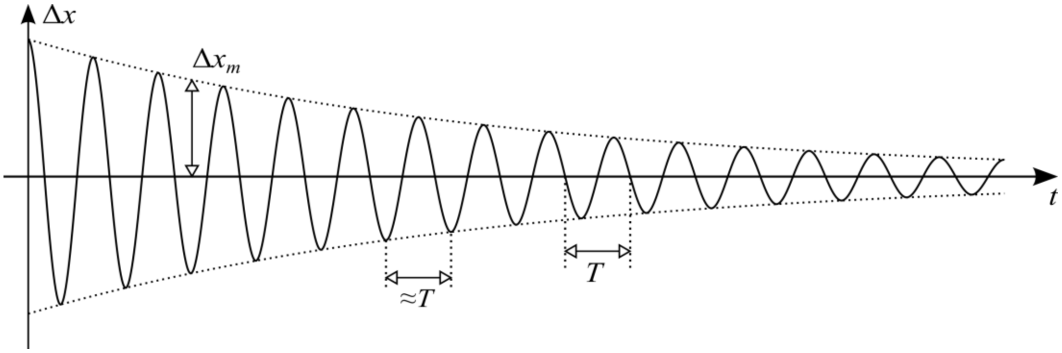

Figure 33.1

Damped harmonic motion.

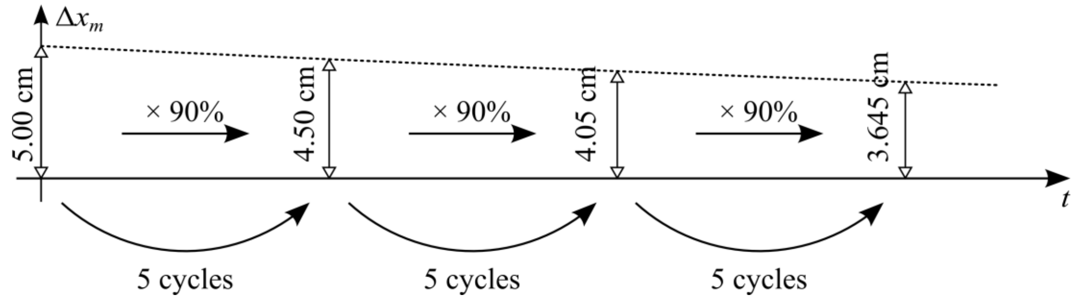

Figure 33.2

Shrinking amplitude of damped harmonic motion. DHM itself is not shown.

The independence of frequency and amplitude is extraordinarily important for the production of music. Consider what would happen if, as the vibration of a guitar string fades away, the frequency (and hence pitch) from the string changed!

The amplitude of DHM also decays in a very specific way. Although the amplitude is only evident where the DHM reaches an extreme displacement, we can imagine the amplitude as a continuous function that forms an envelope around the oscillation, as shown by the dotted lines in Figure 33.1.

> For DHM, over equal intervals of time (or an equal number of cycles) the oscillation amplitude is reduced by a specific constant fraction.

For example, suppose that for some system the amplitude is reduced by 10% for every five cycles. If it were to start with an amplitude of 5 cm, as time goes on the amplitude would change as illustrated in Figure 33.2. For equal-sized steps along the time axis, the amplitude is reduced to a specific fraction of its previous value. This would work for any time interval that you might choose; for a different one (say, three cycles), the reducing fraction would simply be different.

Notice that this means that the oscillation will never die away completely, because the amplitude will always be larger than zero. Of course, it will eventually become imperceptibly small.

Chapter 34. Energy in DHM

As the energy in a damped harmonic motion decreases, the energy must be going elsewhere in order for energy to be conserved. The small additional forces that cause the damping also cause the spring-object system to do work on its surroundings, transferring the energy out. Where does this energy go? There are many possibilities for different situations. Some important examples are heat energy (in the surroundings or in the spring), air turbulence, and, most important for this book, *sound*! If sound is to be "mechanical radiant energy," then its energy must come from somewhere. In order for a vibrating object to create sound, it must be at least a little damped as the sound energy leaves it.

The energy-amplitude proportion can be used to determine how the energy of DHM decreases. Pick two times, t_1 earlier than t_2, and use the variable R for the ratio of the amplitudes at those two times

$$\Delta x_{\max,2} = R\,\Delta x_{\max,1} \quad . \qquad (34.1)$$

For instance, if the two times are two neighboring labeled points in Figure 33.2, then $t_2 - t_1 = 5T$ and $R = 0.9$. Using E to represent energy, proportion 32.1 gives

$$\frac{E_2}{E_1} = \frac{\Delta x_{\max,2}^2}{\Delta x_{\max,1}^2} = R^2 \quad . \qquad (34.2)$$

From this we have two conclusions. First, energy decreases in the same sort of way that amplitude does.

> For DHM, over equal intervals of time (or an equal number of cycles) the total energy is reduced by a specific constant fraction.

Second, for a specific time interval, the reducing fraction for energy is the square of the reducing fraction for amplitude.

Chapter 35. Math for DHM

This chapter lists a few more mathematical details for DHM, which are not required in the rest of the book but which may be of interest.

In order to get damped harmonic motion, as opposed to any other type of damped motion, the damping force must be proportional to the velocity of the moving object,

$$F_{\text{damp}} \propto -v \quad . \tag{35.1}$$

That proportion would have been equally correct without the minus sign, but the minus sign is included as a reminder that the force must be in the opposite direction from the velocity. Otherwise, the force would speed up the motion instead of slowing it down.

As mentioned, this is a good model for many situations, including air resistance. In other common situations, is it not. A block sliding back and forth while supported on a table feels a frictional force from the table which does not obey proportion 35.1, and the resulting motion is not DHM.

We can think of the dotted-line envelope as giving the amplitude as a function of time. The equation for this envelope is an **exponential,**

$$x_m = x_{m0}\,10^{(-t/t_0)} \quad , \tag{35.2}$$

where x_{m0} is the amplitude at the beginning (that is, when $t = 0$) and t_0 is the time interval over which the amplitude decreases to one tenth of its starting value. The 10 is called the **base** of the exponent, and it is an arbitrary choice. If another base is chosen, then t_0 just needs to be adjusted to the time corresponding to the amplitude shrinking by a factor of $1/\text{base}$. But in this book, only the base 10 will be used.