Chapter 13: Partial Differential Equations (PDE's)
(These notes are online as a pdf! Print them before class!)

First of all, this topic is very difficult. And it's new to you. But it's also super cool.

PDE's → means that there is more than one independent variable.

Example: $\nabla^2\phi = 0$ → $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0$ has independent variables x, y, and z.
The dependent variable is ϕ.

As with ODE's, the general procedure is to hope that somebody else can tell you the answer before you even start the problem. But, sometimes you still have to do it yourself.

In physics, there are a zillion relevant PDE's. Example include heat transfer, the equations that describe the motion of waves (e.g., sound, or light, which is a wave of electric field), Schrodinger's equation, momentum conservation in Fluids, etc.

Many of these equations include a ∇^2 somewhere, which always results in derivatives with respect to x, y, and z. In addition, if you want to know the temperature (or whatever), you not only have to specify *where* you want to know ϕ (i.e., at x, y, and z), but often you also have to specify *when* you want to know $\phi(t)$. So, a lot of PDE's have 4 independent variables. Naturally, there are even more kinds of problems than just finding temperature as a function of these 4 variables. You might instead have an equation that, if solved, could tell you pressure as a function of temperature and density.

Separation of Variables

Whenever possible, we solve PDE's by a method called "separation of variables", which is unfortunately **not** anything like the "separation of variables" we used to solve ODE's. For PDE's "separation of variables" is a nickname for a method actually called "Eigenfunction decomposition".

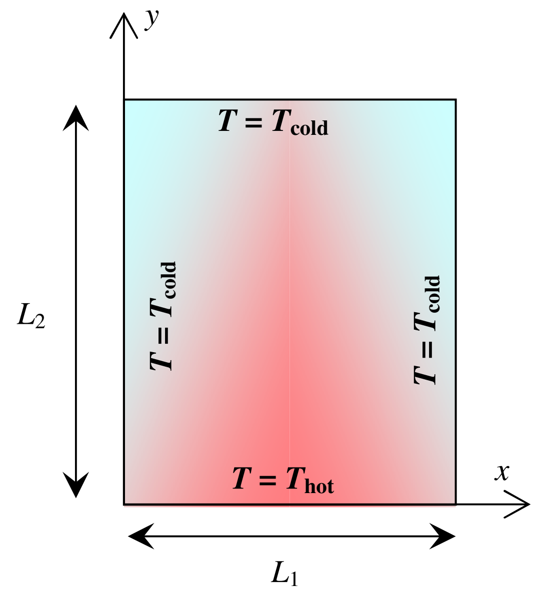
The Heat Equation

Let's start with a simplified form of "the heat equation". This equation is about conduction: how the temperature in one part of an object affects the temperature in other parts. The basic equation is: $k\nabla^2 T=\frac{\partial T}{\partial t}$. This equation assumes that there are no sources of energy embedded in the object, so this version is already somewhat simplified. We will further simplify it by saying that our object is only 2D, and *also* by saying that the temperature profile is "steady" (i.e., that it is not a function of time). With these simplifications, the equation becomes merely $\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$.
Notice that the conductivity of the material, k, cancelled out altogether as a result of making it

"steady". As you imagine, this equation is not that interesting yet. In fact, it still has an infinite number of "answers".

In order to even get started, we have to recognize that there must be some *cause* for the temperature to actually be one way instead of another. For most problems, that means that we need to specify boundary conditions. Looking at this equation, I see two derivatives with respect to x, and two with respect to y. That means that I need to specify a total of **four** boundary conditions before this might even become a real physics problem that I might want to bother with. So, here is the "real" problem I want to solve:

My object is a rectangular 2D sheet of metal (the kind of metal apparently doesn't matter, since k is gone). I will hold the bottom edge against something hot, and the other three edges against something cold. In other words, I will completely specify the temperatures around the edges. As you might imagine, with these boundaries, the temperature of parts of the plate "near" the bottom edge will likely be hotter than those "near" the other edges. That's our goal: to find $T(x, y)$.



Let's write out the boundary conditions explicitly:

$$T(x=0)=T_C$$
$$T(x=L_1)=T_C$$
$$T(y=0)=T_H$$
$$T(y=L_2)=T_C$$

It is customary (but not strictly necessary) to algebraically manipulate both the equation and all the boundary conditions in such a way that the result has no units. This is done so that having created the solution in a generic way, we can apply it to multiple similar situations. It is analogous to solving Analyt I problems symbolically instead of solving them numerically.

Here is the usual way of removing units from equations. We create *new* variables that are linearly related to our original independent and dependent variables:

$$x^* = \frac{x}{L_1} \quad \rightarrow \quad x = x^* L_1$$

$$y^* = \frac{y}{L_1} \quad \rightarrow \quad y = y^* L_1$$

$$N = \frac{L_2}{L_1} \quad \rightarrow \quad L_2 = N L_1$$

$$\boxed{\Theta = \frac{T - T_C}{T_H - T_C} \quad \rightarrow \quad T = (T_H - T_C)\Theta + T_C}$$

Notice that none of the four new variables has any units. *Simple substitution* directly into our basic equation and also into our boundary conditions transforms the equations into these:

$$\frac{\partial^2\Theta}{\partial x^{*2}}+\frac{\partial^2\Theta}{\partial y^{*2}}=0$$

$$\Theta(x^*=0)=0$$
$$\Theta(x^*=1)=0$$
$$\Theta(y^*=0)=1$$
$$\Theta(y^*=N)=0$$

In other words, we will really solve for $\Theta(x^*, y^*)$. Once done, we can convert it back into T, if we care, using the last "purple" equation on the previous page.

At this point, it also customary to admit to ourselves that we are too lazy to bother writing out all the "stars". So, even though it's confusing, we'll write x when we *really* mean x^*. Now, we can start the real work. First, "Separation of Variables": this means that we cross our fingers and hope that the answer is the product of two separate functions, each of which is itself a function of only *one* of the independent variables. Let's call these two functions $X(x)$ and $Y(y)$.
So, we're hoping that $\Theta = X(x)\cdot Y(y)$.

Let's algebraically *substitute* this bit of this crazy hopefulness back into the main equation:

$$\frac{\partial^2\Theta}{\partial x^2}=\frac{\partial^2 X(x)Y(y)}{\partial x^2}=Y(y)\frac{\partial^2 X(x)}{\partial x^2},$$
$$\frac{\partial^2\Theta}{\partial y^2}=\frac{\partial^2 X(x)Y(y)}{\partial y^2}=X(x)\frac{\partial^2 Y(y)}{\partial y^2} \quad \rightarrow$$

$$\frac{\partial^2\Theta}{\partial x^2}+\frac{\partial^2\Theta}{\partial y^2}=Y(y)\frac{\partial^2 X(x)}{\partial x^2}+X(x)\frac{\partial^2 Y(y)}{\partial y^2}=0$$

Then dividing both sides by $X(x)Y(y)$:

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2}+\frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2}=0$$

You might see why they call this "separation of variables"… all the x's are together in one group (both the variable x and the function X), and all the y's are together, too.

If (some group that depends **only on** x) plus (another group that depends **only on** y) adds up to zero, always, then logically it must be true that each group is actually a constant, and one group is the negative of the other. I'll write it this way: $\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = -k^2$ $\quad \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = +k^2$

As you can see, I called the constant k^2, but it has nothing to do with the conductivity k that we saw earlier.

In other words, if my hopes work out, then what I really need to do is solve these *two separate ordinary* differential equations to get X and Y, and then multiply X times Y to find the "total" answer. The decision to call the constant $-k^2$ instead of something more inspirational such as C_1 comes from hindsight, from having done the problem already and noticing that if you called it C_1, then the answer has a $\sqrt{-C_1}$ in it. So, by calling it $-k^2$, we're hoping to make the answer look simpler in the long run.

Let's now solve these two equations separately:

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k^2 \qquad\qquad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = +k^2$$

$$\frac{\partial^2 X(x)}{\partial x^2} = -k^2 X \qquad\qquad \frac{\partial^2 Y}{\partial y^2} = +k^2 Y$$

$$X'' + k^2 X = 0 \qquad\qquad Y'' - k^2 Y = 0$$

From chapter 8, we use the auxiliary equation "D" method to discover:

$$X = A\cos(kx) + B\sin(kx) \qquad\qquad Y = Ce^{ky} + De^{-ky}$$

$$\Theta = \left(A\cos(kx) + B\sin(kx)\right)\left(Ce^{ky} + De^{-ky}\right)$$

We now have 5 unknowns to solve for (A, B, C, D, k), using only our 4 boundary conditions. I hope one of the unknowns magically disappears!

<u>BC #1 (left edge)</u>: $\Theta(x = 0) = 0$ → therefore $\boxed{A = 0}$ → $\Theta = \left(B\sin(kx)\right)\left(Ce^{ky} + De^{-ky}\right)$

<u>BC #2 (right edge)</u>: $\Theta(x = 1) = 0$ → $0 = \left(B\sin(k)\right)\left(Ce^{ky} + De^{-ky}\right)$. The only way for this to happen is if B is always zero, **or** if $\sin(k)$ is always zero. But if $B = 0$, then we have no solution left whatsoever, so let's examine $\sin(k) = 0$. In this case, it must then be true that k is some integer multiple of π. In fact, it can be *any* and *every* integer multiple of π: → $\boxed{k = n\pi}$

A summation appeared!

So, our solution so far is → $\Theta = \sum_{n=1}^{\infty}\left(B_n \sin(n\pi x)\right)\left(Ce^{n\pi y} + De^{-n\pi y}\right)$

<u>BC #3 (top edge)</u>: $\Theta(y = N) = 0$ → $0 = \left(Ce^{kN} + De^{-kN}\right)$, or $D = \frac{-Ce^{+n\pi N}}{e^{-n\pi N}}$.

So, our total answer so far is:
$$\Theta = \sum_{n=1}^{\infty}\left(B_n \sin[n\pi x]\right)\left(Ce^{n\pi y} - C\frac{e^{n\pi N}e^{-n\pi y}}{e^{-n\pi N}}\right)$$

As you see, you'll want to re-write "the answer so far" many, many times!

$$\Theta = \sum_{n=1}^{\infty}\left(B_n \sin[n\pi x]\right)\left(\frac{Ce^{-n\pi N}e^{n\pi y} - Ce^{n\pi N}e^{-n\pi y}}{e^{-n\pi N}}\right)$$

If we review chapter two a little bit, we might recognize that this can be simplified. Specifically,
$$\sinh(x) = \frac{e^{x} - e^{-x}}{2}$$
Plugging this in results in this "simplified" result so far:

$$\Theta = \sum_{n=1}^{\infty}\left(B_n \sin[n\pi x]\cdot E_n \sinh[n\pi(N-y)]\right), \text{ where } E_n = -2Ce^{n\pi N}.$$

To save myself some trouble, I'll call the product $B_n E_n$ to be some new function F_n (this is what eliminates my "5th" unknown, by the way!):

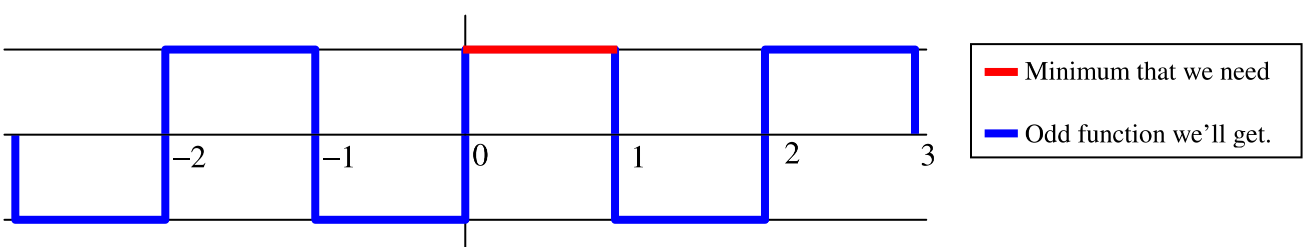
$$\Theta = \sum_{n=1}^{\infty}\left(F_n \sin(n\pi x)\right)\sinh\left(n\pi\left(N-y\right)\right)$$

<u>I have one more BC: BC #4 (bottom edge)</u>: $\Theta(y=0) = 1$

$$1 = \sum_{n=1}^{\infty}\left(F_n \sin(n\pi x)\right)\sinh\left(n\pi\left(N-0\right)\right)$$

This is a lot like the Fourier series problems we've done recently. Noticing that n, π, and N are all just numbers, and again, to save myself some cramping in my hand, I'll once again invent a new letter for this group of constants: $b_n = F_n \sinh(n\pi N)$. In other words, I have:
$$1 = \sum_{n=1}^{\infty}\left(b_n \sin(n\pi x)\right)$$

If this were a Fourier series problem it would have a $1/L$ in front of it. So, this thing on the right hand side is the *odd* Fourier series for $f(x) = 1$ using $L = 1$. To complete the final steps of this problem, let's think of this Fourier series as representing a repeating function. Since it only has sine terms in it, it *better be* an odd repeating function, as opposed to an even one:



Solving this for b_n as we did for all the other Fourier series last week:

$$b_n = \frac{1}{L}\int_{-1}^{0}(-1)\cdot\sin\left(\frac{n\pi x}{L}\right)dx + \frac{1}{L}\int_{0}^{1}(+1)\cdot\sin\left(\frac{n\pi x}{L}\right)dx \;= \frac{2}{n\pi}\left(1-\cos(n\pi)\right)$$

The function $\cos(n\pi)$ is interesting, and might be simplified if we're lucky:
If n is an even number, then $\cos(n\pi) = 1$, and so $b_n = 0$.
If n is an odd number, then $\cos(n\pi) = -1$, and $b_n = \frac{4}{n\pi}$.
I'll take this b_{n}, and plug it back into $b_n = F_n \sinh(n\pi N)$ to find F_n.

So, we have arrived at *some* version of our complete answer:

$$\Theta = \sum_{n=\text{odd}}^{\infty}\left(\frac{4}{n\pi}\frac{\sinh\left(n\pi\left(N-y\right)\right)}{\sinh\left(n\pi N\right)}\sin(n\pi x)\right).$$

Our last step would be to convert n into some other form m that isolates the odd values for us, as we did last week. In this case, odd values of n are generated using $2m - 1$. So, for example,
$m = 1 \rightarrow n = 1$, $m = 2 \rightarrow n = 3$, $m = 3 \rightarrow n = 5$, etc.

$$\boxed{\Theta = \sum_{m=1}^{\infty}\left(\frac{4}{(2m-1)\pi}\frac{\sinh[(2m-1)\pi(N-y^*)]}{\sinh[(2m-1)\pi N]}\sin[(2m-1)\pi x^*]\right)}$$

Also, back-substituting to restore x, y, and T from x^*, y^*, and Θ:

$$\boxed{T = T_C + \sum_{m=1}^{\infty}\left(\frac{4(T_H-T_C)}{(2m-1)\pi}\frac{\sinh[(2m-1)\pi(N-[y/L_1])]}{\sinh[(2m-1)\pi N]}\sin[(2m-1)\pi[x/L_1]]\right)}$$

That's been a lot of work here. It requires us to combine parts from virtually every separate topic that we've studied so far this semester. We used series, complex numbers, chain rules, integrals, differential equations, and Fourier series. The only thing we didn't use was matrices.

See associated Mathematica sheet for generation of this 2D (contour) plot of the solution: