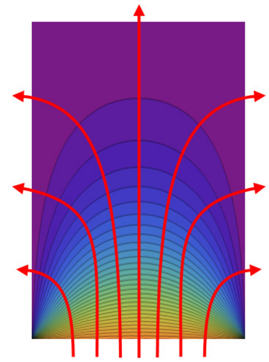
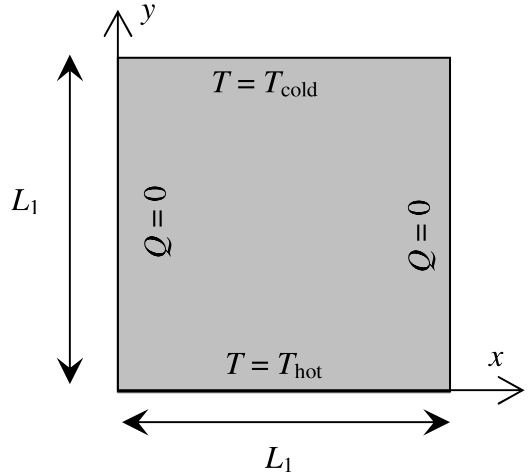
PDE: More Heat Equation with Derivative Boundary Conditions



In our previous problem, we found the temperature $T(x, y)$ in a flat plate, where the bottom edge was forced to be "hot", and the other edges were forced to be "cold". It looked like this. In this sketch, the red arrows show the direction of heat flow (which we didn't solve for, but it would be like some questions from assignment #2). In this case, energy moved from whatever stuff was making the bottom edge hot, into the plate, and then into the ice (or whatever) that was keeping the other edges cold. Over time, that ice would melt if it wasn't replenished, and over time, the hot stuff would cool down if it wasn't maintained, too.

Let's do a few more heat equation problems similar to the previous one. For the first, I'll use a *square* plate ($N = 1$), but I'm going to use different boundary conditions.



This time, the top boundary is still cold, and the bottom boundary is still hot. However, in this problem, the side edges are *insulated*. That means that there is no heat flowing through these edges. This requires us to remember a little bit of physics: since the heat flow Q is proportional to temperature *gradient*, then this boundary condition is the same thing as saying that $\frac{\partial T}{\partial x} = 0$ along both of these edges. Heat can move *vertically* along these edges (i.e., in the y direction, from the bottom to the top of the plate, so $\frac{\partial T}{\partial y} \neq 0$), but *it can't cross out of the plate* in the x direction.

If we had instead insulated the top edge, then the upper boundary condition would be $\frac{\partial T}{\partial y} = 0$ when $y = L_1$. But that's a different problem…

As before, we will non-dimensionalize this whole problem before even "starting" it:

$$\frac{\partial^2 \Theta}{\partial x^{*2}} + \frac{\partial^2 \Theta}{\partial y^{*2}} = 0 \qquad \frac{\partial \Theta}{\partial x^*}\left(x^* = 0\right) = 0 \qquad \Theta\left(y^* = 0\right) = 1$$

$$\frac{\partial \Theta}{\partial x^*}\left(x^* = 1\right) = 0 \qquad \Theta\left(y^* = 1\right) = 0$$

Much of the early work is the same as before, so we'll steal results from ourselves… we can "borrow" everything that we did up until we started using the boundary conditions!

Recall that when we started our "Separation of Variables", we invented new functions $X(x)$ and $Y(y)$, for which we hoped that eventually $\Theta(x, y) = X(x)\cdot Y(y)$. By substitution, we discovered that if Θ followed this pattern, then instead of solving one big PDE we could instead solve two ODE's and than join them together. They were these:

$$\frac{\partial^2 X(x)}{\partial^2 x} + k^2 X(x) = 0 \qquad \text{and} \qquad \frac{\partial^2 Y(y)}{\partial^2 y} - k^2 Y(y) = 0$$

These were both second-order ODEs with constant coefficients, and we chose to write the solution for X in terms of sines and cosines, but we chose to write the solution for Y in terms of exponentials. Specifically, we found that:

$$X = A\cos(kx) + B\sin(kx) \qquad Y = Ce^{ky} + De^{-ky}$$

$$\boxed{\Theta = \left(A\cos(kx) + B\sin(kx)\right)\left(Ce^{ky} + De^{-ky}\right)}$$

…stolen from last time!

Because of our new boundary condition, we might also need the x derivatives of Θ later, so:

$$\frac{\partial\Theta}{\partial x} = \left(kA\sin(kx) - kB\cos(kx)\right)Y(y)$$

Using BC #1 (left edge): $\frac{\partial\Theta}{\partial x}(x=0) = 0 \rightarrow$ $0 = \left(kA\sin(k0) - kB\cos(k0)\right)Y(y)$
$0 = kB$

Well, either k or B could be zero. If we choose $k = 0$, we end up with nothing at all, because k is inside both the sine and cosine parts of the answer. So, I guess this is telling us that $\boxed{B = 0.}$

Using BC #2 (right edge): $\frac{\partial\Theta}{\partial x}(x=1) = 0 \rightarrow$ $0 = kA\sin(k\cdot 1)Y(y)$
$\boxed{k = n\pi}$

Well, that's pretty similar to what we had last time. So far, then, we seem to have this:

$$\boxed{\Theta = \sum_{n=0}^{\infty} A_n \cos(n\pi x)\left(Ce^{ky} + De^{-ky}\right)}$$

← keep re-writing summary so far!

Note that unlike last time, *this* infinite series starts at *zero*, since we now have cosine terms, but in the problem from the previous class, we only started at $n = 1$, since we had only sine terms in that other problem. This creates a subtle problem for us… if $n = 0$, then $k = 0$, and if $k = 0$, then our original two ODE's actually looked like this:

$$\frac{\partial^2 X(x)}{\partial^2 x} = 0 \qquad \text{and} \qquad \frac{\partial^2 Y(y)}{\partial^2 y} = 0$$

But, for *these* ODE's the solutions are NOT sines and cosines… each of these is actually an *easier* "separate and integrate" ODE! Specifically, when $k = n = 0$, then:

$X_{k=0} = C_1 x + C_2,$ and $Y_{k=0} = C_3 y + C_4.$

Therefore, our total solution so far is really: $\Theta = \sum_{n=0}^{\infty}(\Theta_n) = \sum_{n=1}^{\infty}(\Theta_n) + \Theta_0$

So more specifically: $\Theta = \sum_{n=1}^{\infty} A_n \cos(n\pi x)\left(Ce^{n\pi y} + De^{-n\pi y}\right) + (C_1 x + C_2)(C_3 y + C_4)$

Oh no! Now we have 4 more *new* constants to solve for! Also, I hope this doesn't screw up the boundary conditions we've already done! Let's double check the left and right edges, just to be safe.

Redoing BC #1 (left): $\left.\frac{\partial \Theta}{\partial x}\right|_{x=0} = 0 = (0)\left(Ce^{n\pi y} + De^{-n\pi y}\right) + (C_1)(C_3 y + C_4)$ → $C_1 = 0$

Solution so far: $\Theta = \sum_{n=1}^{\infty} A_n \cos(n\pi x)\left(Ce^{n\pi y} + De^{-n\pi y}\right) + (C_2)(C_3 y + C_4)$

Combining annoying constants: $\Theta = \sum_{n=1}^{\infty} A_n \cos(n\pi x)\left(Ce^{n\pi y} + De^{-n\pi y}\right) + (C_5 y + C_6)$

i.e., let $C_5 = C_2 C_3$ and $C_6 = C_2 C_4$

Rechecking BC #2 (right): $\left.\frac{\partial \Theta}{\partial x}\right|_{x=1} = 0 = (0)\left(Ce^{n\pi y} + De^{-n\pi y}\right) + 0$ is already (or still) OK!

BC #3 (top edge): $\left.\Theta\right|_{y=1} = 0 = \left(A_n \cos(n\pi x)\right)\left(Ce^{n\pi} + De^{-n\pi}\right) + C_5 + C_6$

The C_5 and C_6 worry me a little. How can we tell them apart? What are the possibilities?

a) If $C_5 = C_6 = 0$, then this boundary condition is the same as what we did last time, but the $n = 0$ part of the solution will have completely disappeared.. so no thanks!
b) If $C_5 + C_6 = 0$, then this boundary condition is the same as what we did last time.
c) If $C_5 + C_6 \neq 0$, then $A_n\cos(n\pi x)$ = some non-zero number for all values of x. This is just not possible.

So, of these options, version b) is the most general one. In other words, $C_6 = -C_5$. Also, we get to use our solution from the other day, too! That time, C and D turned into a hyperbolic sine that depended on N, and we regrouped A and C into a new constant "F" This time, $N = 1$, so our solution so far is:

$$\Theta = \sum_{n=1}^{\infty} F_n \cos(n\pi x)\sinh\left(n\pi(1-y)\right) + C_5(y-1)$$

Bottom edge: BC #4: $\Theta(y=0) = 1$

$$1 = \sum_{n=1}^{\infty} F_n \cos(n\pi x)\sinh\left(n\pi(1-0)\right) + C_5(0-1)$$

$$1 + C_5 = \sum_{n=1}^{\infty} F_n \cos(n\pi x)\sinh(n\pi)$$

We've seen this kind of thing before. Let's invent a new constant to group some others:

$$a_n = F_n \sinh(n\pi) \qquad \text{Leading to:}$$

$$1 + C_5 = \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

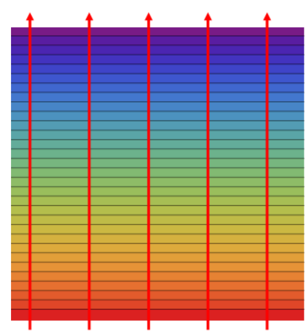
We want to solve this for a_n. This is a Fourier series problem again:

$$a_n = 2\int_0^1 (1 + C_5)\cdot \cos(n\pi x)dx = \{Mathematica\} = 0 \text{ for all values of } n!!!!$$

Well, that's unexpected good news. So, the bottom BC #4 reduces to: $1 + C_5 = 0$, $\boxed{C_5 = -1}$

Plugging this in so far, we have: $\boxed{\Theta = (1 - y) + \sum_{n=1}^{\infty} \frac{(0)\sinh(n\pi(1-y))}{\sinh(n\pi)} \cos(n\pi x)}$

Or just: $\boxed{\Theta = 1 - y}$



Well, that's nuts. That was way too much work for such a simple answer. This answer says that the temperature is not really a function of x after all, and that the table gets cooler as you go from the bottom to the top. But it satisfies the main equation and all the boundary conditions!

Here's the result, which seems obvious when we draw it:

Non-Uniform Temperature along Bottom Edge

Let's do a different, but similar problem. This one has the same BC's for left, right, and top (being no heat, no heat, and no temperature). But now, the lower edge temperature is $\Theta_{\text{bottom}} = x$. In other words, the bottom is cold on the left, hot on the right, and rising steadily in between. All we're changing is BC #4, so we only need to go back to just before we applied BC#4 (near the bottom of page 3…). This time:

$$\Theta(y = 0) = x$$

$$x = \sum_{n=1}^{\infty} F_n \cos(n\pi x)\sinh(n\pi(1-0)) + C_5(0-1)$$

$$x + C_5 = \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

For this Fourier series: $a_n = 2\int_0^1 (x + C_5)\cos(n\pi x)dx$ → $a_n = 0$ when n is even,

→ $a_n = \dfrac{-4}{n^2\pi^2}$ when n is odd.

This suggests that we want to count n = 1, 3, 5, 7. But in fact, we usually count n = 1, 2, 3, 4. Also, C_5 doesn't even matter to finding a_n!

To isolate the odd values, we use the same old trick as always: $a_n = \dfrac{-4}{(2n-1)^2\pi^2}$.

Our solution so far: $$\Theta = C_5\left(y-1\right) + \sum_{n=1}^{\infty}\frac{-4}{\left(2n-1\right)^2\pi^2}\frac{\cos((2n-1)\pi x)\sinh((2n-1)\pi(1-y))}{\sinh((2n-1)\pi)}$$

We still need to find a number for C_5… I notice that $\Theta(0, 0) = 0$ (bottom left corner). So:

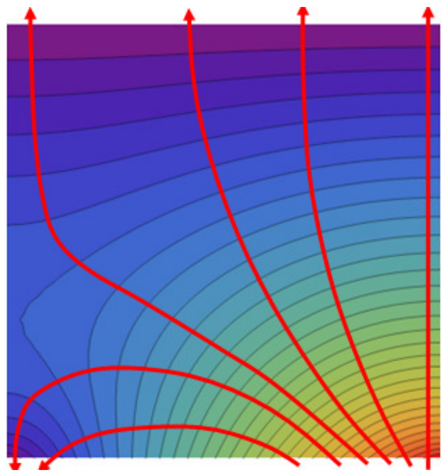
$$0 = C_5(0-1) + \sum_{n=1}^{\infty}\frac{-4}{(2n-1)^2\pi^2}\frac{\cos[(2n-1)\pi\cdot 0]\sinh[(2n-1)\pi\cdot(1-0)]}{\sinh[(2n-1)\pi]}$$

$$0 = -C_5 + \sum_{n=1}^{\infty}\frac{-4}{(2n-1)^2\pi^2}\frac{1\cdot\sinh[(2n-1)\pi]}{\sinh[(2n-1)\pi]}$$

$$0 = -C_5 + \sum_{n=1}^{\infty}\frac{-4}{(2n-1)^2\pi^2}$$

$$0 = -C_5 - \frac{1}{2} \quad \{\text{infinite sum done in Mathematica}\}$$

$$C_5 = -\frac{1}{2}$$



Yay! We're done! Here is the full answer:

$$\Theta = \frac{1-y^*}{2} + \sum_{n=1}^{\infty}\frac{-4}{(2n-1)^2\pi^2}\frac{\cos((2n-1)\pi x^*)\sinh((2n-1)\pi(1-y^*))}{\sinh((2n-1)\pi)}$$

This is also plotted in Mathematica online (available on course demos page). Notice the lack of *slope* along the side edges, and the linear profile along the bottom edge, in agreement with our boundary conditions. The Mathematica version is 3D "rotatable" to see this.

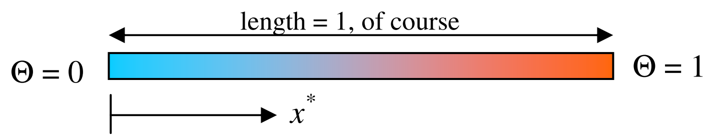
Unsteady Heat Equation

One version of the heat equation is: $\nabla^2 T = \dfrac{1}{\alpha^2}\dfrac{\partial T}{\partial t}$ where α is a material property related to the conductivity. We will non-dimensionalize this as before. The only new feature now is that we need to invent a dimensionless time t^*: $t^* = \left(\dfrac{\alpha^2}{L^2}\right)t$. Making this substitution, and again being too lazy to write the asterisk after every x, y, and t, the heat equation becomes: $\nabla^2\Theta = \dfrac{\partial\Theta}{\partial t}$. Since ∇^2 is really a function of all 3 spatial dimensions x, y, and z, this problem is way too hard for us. So to simplify it, we'll look at a thin stick, which has only one spatial dimension: *x*. So, this is still a "2D" problem, with independent variables *x** and *t**: $\frac{\partial^2\Theta}{\partial x^2}-\frac{\partial\Theta}{\partial t}=0$

This differs in two ways from our prior examples: here, one of the derivatives is only 1st order, and here, one of the derivatives is negative.

Counting the derivatives, I'll need three "boundary" conditions: two relating to *x*, and only one about *t*. Here they are:

The left edge ($x = 0$) is held at $\Theta = 0$ ("cold"),
The right edge ($x = 1$) is held at $\Theta = 1$ ("hot").
The third "initial" condition is that $\Theta = 0$ ("cold") *everywhere* along the bar *when we start*.



In other words, it starts off all cold, but it heats up over time as heat leaks into the bar from the right hand edge.

We've learned some tricks about such problems from the last two examples. We will assume that the answer is comprised of two functions *X* and *T* that are multiplied together (Separation of Variables, or "Eigenfunction Decomposition"), and we will add on an extra bit like we did in the previous problem for $n = 0$ to account for the "steady state" solution:

$$\boxed{\Theta = X(x)\cdot T(t)+\ \Theta_{ss}}$$

Adding the steady state solution is a lot like what we did with non-homogenous ODE problems, where we assumed that $y_{\text{complete}} = y_C + y_{\text{Particular}}$. So, $\Theta_c = X(x)\cdot T(t)$, and $\Theta = \Theta_c + \Theta_{ss}$.

If a steady state temperature distribution is reached, then by definition, the temperature is done changing. Let's deal with this Θ_{SS} part first. So, as $t \rightarrow \infty$, $\frac{\partial\Theta}{\partial t}=0$. So, when $t = \infty$, the heat equation itself simplifies to $\frac{\partial^2\Theta_{ss}}{\partial x^2}-\frac{\partial\Theta_{ss}}{\partial t}=0$, or even better: $\frac{\partial^2\Theta_{ss}}{\partial x^2}=0$

Integrating twice, this becomes a solution for Θ_{ss}: $\Theta_{ss} = C_1x + C_2$.
Since $\Theta(x = 0) = 0$, then $C_2 = 0$.
Since $\Theta(x = 1) = 1$, then $C_1 = 1$.
This simplifies our stead state portion to this: $\boxed{\Theta_{ss}=x}$ $\rightarrow$ $\boxed{\Theta = X(x^*)T(t^*)+x^*}$

Now, let's work on Θ_C:

$$\Theta_C = X(x)T(t)$$

$$\frac{\partial\Theta_C}{\partial x}=X'T \qquad \frac{\partial\Theta_C}{\partial t}=XT'$$

$$\frac{\partial^2\Theta_C}{\partial x^2}=X''T$$

The heat equation therefore becomes: $X''T - XT' = 0$

Divide by *XT*:

$$\frac{X''}{X} - \frac{T'}{T} = 0$$

As before, this is only possible if each group is separately constant:

$$\frac{X''}{X} = -k^2 \qquad -\frac{T'}{T} = +k^2$$

$$X'' + k^2 X = 0 \qquad T' + k^2 T = 0$$

From our chapter on ODE's, the answers to these two equations are:

$$X = A\cos(kx) + B\sin(kx) \qquad T = Ce^{-k^2 t}$$

Since I'll be multiplying these expressions, the constant *C* is redundant, and we'll work it into *A* and *B* as we've done with all the other examples so far:

$$\Theta_c = [A\cos(kx) + B\sin(kx)]e^{-k^2 t}$$

Combining Θ_C and Θ_{ss}: $$\boxed{\Theta = \Theta_c + \Theta_{ss} = [A\cos(kx) + B\sin(kx)]e^{-k^2 t} + x}$$

Now, we apply the boundary conditions:

<u>BC #1 (left)</u>: $\Theta(x = 0) = 0$ → $0 = \left(A\cos(0) + B\sin(0)\right)e^{-k^2 t} + 0$ (this last "0" is Θ_{ss})

Left edge is cold…

→ $\boxed{A = 0}$

<u>BC #2 (right)</u>: $\Theta(x = 1) = 1$ →

Right edge is hot…

$$1 = \left(B\sin(k)\right)e^{-k^2 t} + 1 \quad \text{(this last bit is } \Theta_{ss}\text{)}$$

$$0 = \left(B\sin(k)\right)e^{-k^2 t}$$ Since B $\neq 0$, it must be true that:

$\boxed{k = n\pi}$ Well, we've seen that before!

Our answer so far: $$\boxed{\Theta = \sum_{n=1}^{\infty} B_n \sin(n\pi x)e^{-n^2\pi^2 t} + x}$$

<u>Finally, we apply the *initial* condition: BC #3</u>: $\Theta(t = 0) = 0$ →

$$0 = \sum_{n=1}^{\infty} B_n \sin(n\pi x)e^{-n^2\pi^2 0} + x$$

$$-x = \sum_{n=1}^{\infty} B_n \sin(n\pi x)$$

Note that *n* starts counting at 1, not 0, because this is a sine series.

This is a regular old Fourier series that we've gotten good at:

$$B_n = 2\int_0^1 (-x)\sin(n\pi x)dx \qquad \text{(or equivalently , } B_n = 1\int_{-1}^{+1}(-x)\sin(n\pi x)dx\text{)}$$

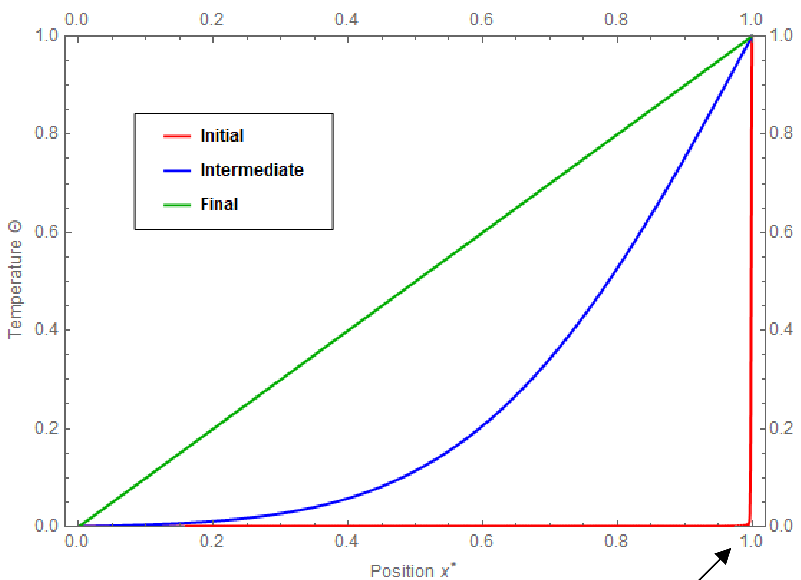
Mathematica says: $B_n = \dfrac{2(-1)^n}{n\pi}$.

So, we're just about done: $$\Theta = \sum_{n=1}^{\infty}\frac{2(-1)^n}{n\pi}\sin(n\pi x^*)e^{-n^2\pi^2 t^*} + x^*$$

Putting the units back in: $x^* = x/L$ and $t^* = \left(\dfrac{\alpha^2}{L^2}\right)t$:

$$T = T_C + (T_H - T_C)\left[\sum_{n=1}^{\infty}\frac{2(-1)^n}{n\pi}\sin\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\pi^2\alpha^2}{L^2}t} + \frac{x}{L}\right]$$

Some plots of Θ are shown in Mathematica online. Each separate line represents a time*, and the vertical axis is temperature.



As before, there's an "infinite" slope here, because we contradicted ourselves in the problem statement. Specifically, we said that:

a) the entire stick is cold at the beginning, and
b) the right hand side is *always* hot.